# FIRST STEPS IN TOPOLOGICAL TRANSFORMATION GROUPS

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Abstract. The theory of topological transformation groups combines algebra with topology in a beautiful way. This short course gives an introduction to some of the very basic definitions concerning topological transformation groups and a few simple results. All the omitted proofs may be found in any book about transformation groups, the book by Kawakubo [Kaw91] being one of them. Due to the limited amount of time it is impossible to provide more than only a short glimpse of the surface of this theory, but it might give an insight into the interplay of algebraic and topological concepts.

## 1. Introduction

A map $f\colon X \to X$ from a set $X$ to itself is called a *transformation* of the set $X$. It is not our interest to study the properties of a single transformation but rather to study families of transformations of a set $X$ at the same time. In particular we are interested in studying sets of transformations of a set $X$ which exhibit a group structure. If $G$ is a set of transformations of $X$ which has a group structure then we call $G$ a *transformation group* of $X$ and $X$ is called a *$G$-set*. We say also that $G$ acts on $X$. If $X$ has some additional mathematical structure then we usually require the transformations to preserve this structure.

We use transformation groups to study the structure and symmetries of mathematical objects like sets, rings, fields, topological spaces and differentiable manifolds.

Galois theory is a well known example which shows the importance of considering a set of transformations as a group. The following theorem is part of the main result of Galois theory (see for example [Lan99, p. 262]).

**Theorem.** *Let $K$ be a finite Galois extension of a field $k$, with Galois group $G = G(K/k)$.*[1] *Then there is a bijection between the sets of subfields $E$ of $K$ containing $k$, and the set of subgroups $H$ of $G$, given by $E = K^H$.*

As a consequence of this theorem it is enough to calculate the Galois group $G(K/k)$ of a finite Galois extension $K$ of $k$ and then determine the subgroup structure of this group in order to get a complete and exhaustive picture of possible subfields $k \subset E \subset K$ and their relation to each other. Since the Galois group of a finite Galois extension is finite this reduces the usual infinite problem to a finite problem.

## 2. Topological Groups

We do not want to restrict ourself to transformation groups in the purely algebraic sense and introduce the concept of topological groups. We define the category of topological groups.

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[1]Recall that the an extension $K$ of a field $k$ is called *Galois* if it is normal and separable. The *Galois group* $G(K/k)$ of such an extension is the group of automorphisms of $K$ over $k$.

**Definition 1** (Topological Group)**.** A $T_1$-space[2] $G$ which is at the same time a group is called a *topological group* if the maps

$$\mu\colon G \times G \to G, (g,h) \mapsto gh$$

and

$$\imath\colon G \to G, g \mapsto g^{-1}$$

are both continuous.

The $T_1$ condition is included in the above definition in order to avoid some pathological cases of topologies. This restriction does not impose any problems because most of the topological groups encountered in everydays life are locally compact or even compact groups.[3] It can be shown that for topological groups the $T_1$ condition implies the $T_2$, that is, every topological group is Hausdorff.

Note that any group $G$ becomes a topological group if it is equipped with the discrete topology. Therefore the concept of a topological group extends the concept of groups in the purely algebraic sense.

If $G$ is a topological group, then any subspace $H$ of $G$ inherits the $T_1$ property. Thus, if $H$ is closed under the multiplication of $G$ it follows that $H$ is a topological group under the multiplication of $G$, too.

**Definition 2.** Let $G$ be a topological group. A *subgroup* $H$ of the topological group $G$ is a subgroup of $G$ in the algebraic sense equipped with the subspace topology.

In a similar spirit we define what we mean by a homomorphism of topological groups.

**Definition 3.** A *homomorphism* $f\colon G \to G'$ of topological groups is a group homomorphism in the algerabic sense which is continuous. The homomorphism $f$ is said to be an *isomorphism* of topological groups if $f^{-1}\colon G' \to G$ exists and is continuous. Two topological groups $G$ and $G'$ are said to be *isomorphic* if there exists an isomorphism $f\colon G \to G'$ of topological groups.

Thus an isomorphism of topological groups is an isomorphism of groups in the purely algebraic sense which is also a homeomorphism. Note that it is *not* enough for a homomorphism of topological groups to be both a monomorphism and an epimorphism in order to be an isomorphism. Thus being isomorphic as topological groups is a stronger requirement than being isomorphic in the purely algebraic sense.

For example let $G$ be the group of real numbers $\mathbb{R}$ under addition and equipped with the discrete topology and let $G'$ be the group of real numbers under addition equipped with the standard topology of the real line. Both are topological groups and the identity

$$\mathrm{id}\colon G \to G'$$

is a bijective homomorphism of topological groups. But it is not an isomorphism of topological groups since the inverse $\mathrm{id}^{-1} = \mathrm{id}\colon G' \to G$ is not continuous. In fact despite $G$ and $G'$ are by construction isomorphic in the algebraic sense they cannot be isomorphic as topological groups since their topologies are essential different (for example the topology of $G$ is discrete whereas $G'$ is path connected).

When speaking of topological groups and their subgroups we conveniently attach topological terminology to them. That is, we say that a topological group is compact

[2]A topological space $X$ is said to be $T_1$ if all singleton subsets of $X$ are closed in $X$.

[3]In this text we use the convention that compactness shall *always* imply the Hausdorff condition.

or locally compact if the underlying topological space is compact or locally compact. Similarly we say that a subgroup $H$ is open (or closed) in $G$, if $H$ is a open (or closed) subset of $G$ seen from the topological point of view. The only exception to this rule is that by a normal subgroup we mean a subgroup which is a normal subgroup in the algebraic sense.

If $G$ is a topological group and $H$ a normal subgroup of $G$, then we can form in the usual way the quotient group $G/H$ which consist of left cosets $gH$, $g \in G$. Denote by $\pi$ the canonical projection homomorphism

$$\pi\colon\, G \to G/H, g \mapsto gH.$$

We topologize the set $G/H$ by requiring that $U \subset G/H$ is open if and only if $\pi^{-1}(U)$ is an open subset of $G$. That is, we equip $G/H$ with the quotient topology. With this topology on $G/H$ the projection $\pi$, becomes an open map, and since it is by construction surjective it is a quotient map. It follows that $G/H$ is $T_1$ if and only if $H$ is a closed subgroup of $G$.

**Definition 4.** Let $G$ be a topological group and $H$ a closed subgroup of $G$. Then $G/H$ denotes the *quotient group* which is the topological group obtained from the group of left cosets equipped with the quotient topology.

## 3. Examples of Topological Groups

The real line $\mathbb{R}$ equipped with the standard topology is a topological group under addition since

$$(x, y) \mapsto x + y \qquad \text{and} \qquad x \mapsto -x$$

are continuous maps. Similarly $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$ and $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ equipped with the standard topology is a topological group under the multiplication.

If $n > 0$ is a positive integer the real vector space $\mathbb{R}^n$ equipped with the standard topology is a topological group under the usual vector addition.

The General Linear Group $\mathrm{GL}_n(\mathbb{R})$ can be seen as a subset of $\mathbb{R}^{n^2}$, the latter equipped again with the standard topology. We give $\mathrm{GL}_n(\mathbb{R})$ the subspace topology. With respect to this topology the matrix multiplication and matrix inversion are continuous.[4] It follows that $\mathrm{GL}_n(\mathbb{R})$ is a topological group. It is one of the classical matrix groups. It has amongst others the following closed subgroups:

$$\begin{aligned} \mathrm{O}_n &:= \{A \in \mathrm{GL}_n(\mathbb{R}) : A\,{}^t\!A = I\}, \\ \mathrm{SL}_n(\mathbb{R}) &:= \{A \in \mathrm{GL}_n(\mathbb{R}) : \det A = 1\}, \\ \mathrm{SO}_n &:= \mathrm{O}_n \cap \mathrm{SL}_n(\mathbb{R}). \end{aligned}$$

In similar way $\mathrm{GL}_n(\mathbb{C})$ is made into a topological group (seen as a subspace of $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$) and it has amongst others the following closed subgroups:

$$\begin{aligned} \mathrm{U}_n &:= \{A \in \mathrm{GL}_n(\mathbb{C}) : A\,{}^t\!\bar{A} = I\}, \\ \mathrm{SL}_n(\mathbb{C}) &:= \{A \in \mathrm{GL}_n(\mathbb{C}) : \det A = 1\}, \\ \mathrm{SU}_n &:= \mathrm{U}_n \cap \mathrm{SL}_n(\mathbb{C}). \end{aligned}$$

Note that all the above examples of topological groups are locally compact groups, some are even compact groups (which?).

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[4] The matrix multiplication can be seen to be continuous from the product formula for matrix multiplication and the matrix inversion can be seen to be continuous by the Cramer's rule for the matrix inversion.

## 4. Topological Transformation Groups

In the following $G$ denotes a topological group with the identity element denoted by $e$. We define the category of $G$-spaces.

**Definition 5** ($G$-space)**.** Let $X$ be a topological space. A continuous map

$$\Phi\colon G \times X \to X$$

is called an *(continuous) action* of $G$ on $X$ if it satisfies

(1) $\Phi(e, x) = x$ for all $x \in X$ and
(2) $\Phi(g_1, \Phi(g_2, x)) = \Phi(g_1 g_2, x)$ for all $x \in X$ and $g_1, g_2 \in G$.

A *$G$-space* $X = (X, G, \Phi)$ consists of a topological space $X$, a topological group $G$ and a continuous action $\Phi$. In this case $G$ is called a *topological transformation group* on $X$

If there is no danger of confusion (and this is the case if the reader is awake) we write $gx$ instead of $\Phi(g, x)$.

Any topological group $G$ can act trivially on any space $X$, that is the action of $G$ on $X$ is given by $\Phi(g, x) := x$ for any $x \in X$, $g \in G$. Thus $G$-spaces exist and we are not talking about the empty set. A less trivial example is the usual action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$ which makes $\mathbb{R}^n$ into a $\mathrm{GL}_n(\mathbb{R})$-space. As a subgroup of $\mathrm{GL}_n(\mathbb{R})$ the orthogonal group $\mathrm{O}_n$ acts on $\mathbb{R}^n$ leaving the length of vectors unchanged. Thus $\mathrm{O}_n$ acts on the unit disc $D^n$ and on the unit sphere $S^{n-1}$. As a last example consider a subgroup $H$ of $G$. Then

$$\Phi\colon G \times G/H \to G/H, (g', gH) \mapsto g'gH$$

defines in a natural way a $G$-action on the set of left cosets $G/H$. $G$-spaces of such type are called *homogeneous* $G$-spaces. If $H$ is a closed subgroup of $G$ then the homogeneous space $G/H$ will be Hausdorff.

**Definition 6.** Let $X$ and $Y$ be $G$-spaces. An *equivariant* map

$$f\colon X \to Y$$

is a contiuous map such that $f(gx) = gf(x)$ for every $x \in X$. An equivariant map is also called a *$G$-map*.

For a fixed topological group $G$ the class of all $G$-spaces together with $G$-maps between them form a category.

Usually one assumes that the topological spaces are nice spaces which in our case shall mean that they are Hausdorff. This is not a very restrictive assumption and from here on we shall assume that all the spaces in question are nice.

## 5. Fixed Points, Isotropy Groups, Orbits and Orbit Spaces

In this section we define some very basic objects. Throughout this section $G$ is a topological group and $X$ a $G$-space.

Given a subset $H$ of $G$ – and usually $H$ will be a subgroup – we define the *fixed point set* $X^H$ to be

$$X^H := \{x \in X : gx = x \text{ for all } g \in H\}.$$

If $H = \{g\}$ is a singleton set it is customary to write $X^g$ instead of $X^H$. Straight from the definition we have the equality

$$X^H = \bigcap_{g \in H} X^g$$

and for a subset $K \subset G$ the implication

$$H \subset K \Rightarrow X^K \subset X^H.$$

As a consequence from $X$ being a Hausdorff (as we have agreed above) we get the following result:

**Proposition 7.** *For any subset $H$ the fixed point set $X^H$ is a closed subset of $X$.*

For any $x \in X$ we can consider the set

$$G_x := \{g \in G : gx = x\}.$$

This set is in fact a subgroup of $G$ and it is called the *isotropy group* at $x$.

As a consequence from $X$ being a $T_1$-space (Hausdorff implies $T_1$!) we get the following result:

**Proposition 8.** *The isotropy group $G_x$ at any point $x$ of $X$ is a closed subgroup of $G$.*

We say that two points $x, y \in X$ have the same *isotropy type* if $G_x$ and $G_y$ are conjugate subgroups in $G$. For any $x \in X$ it follows that $x$ and $gx$ have the same isotropy type for any $g \in G$. More precisely we have

**Proposition 9.** *For any $x \in X$ and $g \in G$, we have*

$$G_{gx} = gG_xg^{-1}.$$

For any $x \in X$ we define the *orbit* $G(x)$ of $x$ under $G$ to be the set

$$G(x) := \{gx : g \in G\}.$$

**Proposition 10.** *For any $x, y \in X$, either $G(x) = G(y)$ or $G(x) \cap G(y) = \emptyset$.*

Thus the orbits of $X$ form a partition of $X$. We say that $x$ and $y$ *belong to the same orbit* if $G(x) = G(y)$.[5] This is clearly an equivalence relation on $X$ and we can define the *orbit space* $X/G$ to be the quotient space under this relation. By definition we have then that the canonical projection

$$\pi\colon X \to X/G, x \mapsto G(x)$$

onto the orbit space is an open map.

Be aware that even if $X$ is a topological space with very nice properties, the quotient space $X/G$ might not inherit them in general. For example one can easily construct a continuous action of $\mathbb{R}$ on the torus $T = S^1 \times S^1$ such that $T/\mathbb{R}$ is a infinite set but has the trivial topology (see [Kaw91, p. 37]).

Still we are able to describe the structure of the orbit:

**Proposition 11.** *For any $x \in X$ there exists a $G$-homeomorphism*

$$f\colon G(x) \to G/G_x.$$

## 6. Types of $G$-Actions

Let $G$ be a group and $X$ a $G$-space. There are several basic ways to classify the action of $G$ on $X$. The action of $G$ on $X$ is said to be:

(1) *trivial* if $G_x = G$ for every $x \in X$;
(2) *free* if $G_x = \{e\}$ is the trivial group for every $x \in X$;
(3) *semi-free* if $G_x = G$ or $G_x = \{e\}$ for all $x \in X$;
(4) *transitive* if $X = G(x)$ for some (and therefore all) $x \in X$;
(5) *effective* if $\bigcap_{x\in X} G_x = \{e\}$.

Note that these types do not necessarily exclude each other. For example: A trivial action is semi-free. And a free action is always semi-free and effective.

[5] Note that a necessary condition for two points to belong to the same orbit is that they have the same orbit type.

## 7. Compact Transformation Groups

Amongst the transformation groups the compact transformation groups have an important role. Every finite group is necessarily a compact topological group. The unit circle $S^1$ and the group of $p$-adic integers $\mathcal{Z}_p$ are other examples of compact groups.[6] Finally the classical matrix groups $\mathrm{O}_n$, $\mathrm{SO}_n$, $\mathrm{U}_n$ and $\mathrm{SU}_n$ are compact groups.

It turns out that the theory of *compact* transformation groups has its own very special flavor. The classical book by Berdon on compact transformation groups [Bre72] is solely devoted to this specific area.

The next result shall give a first glimpse on what nice results one can obtain for compact transformation groups. It is one of the first basic conclusions one may make in this theory (see [Kaw91, p. 38]).

**Proposition 12.** *Let $G$ be a compact topological group and $X$ a $G$-space. Then the following holds.*

(1) *$\pi\colon X \to X/G$ is a closed and proper[7] map.*
(2) *$X/G$ is Hausdorff.*
(3) *$X$ is compact $\Leftrightarrow$ $X/G$ is compact.*
(4) *$X$ is locally compact $\Leftrightarrow$ $X/G$ is locally compact.*

Compare this result with the example given on page 5. There $T$ is a compact manifold, but the canonical projection onto the orbit space $\pi\colon T \to T/\mathbb{R}$ is neither closed nor proper and the orbit space $X/G$ is neither Hausdorff nor compact nor locally compact (the latter two even if one does not assume that compactness should imply the Hausdorff condition as we have agreed).

## References

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[6]The group of $p$-adic integers is as the inverse limit of finite – and thus compact – groups itself compact.

[7]Recall that a continuous map is said to be *proper* if the preimage of any compact set is compact.