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Mathematics

P.A. Smith Theory
for p -Adic Transformation Groups

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<p>In 1960 C.T. Yang published an article on the action of the group of p-adic integers on homology n-manifolds. In this article Yang developed a modified version of Smith Theory. This thesis describes in detail the construction process of this modified version of Smith Theory. The comprehensibility of Yang's construction is improved, some mistakes in his exposition are corrected and some minor improvements are achieved.</p>			
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To
Helge

If you only walk on sunny days,
you will never reach your destination.
Buddhist monk

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Introduction

1. The Hilbert–Smith Conjecture

The Hilbert–Smith conjecture – which is a generalized version of Hilbert’s Fifth Problem, see [Hil00] – states that among the locally compact groups only Lie groups can act effectively on a finite dimensional manifold. More precisely:

Conjecture (Hilbert–Smith). *Let M be a connected manifold. Assume that G is a locally compact group acting effectively on M . Then G can be given the structure of a Lie group.*

Though the original problem has been solved affirmative the Hilbert–Smith conjecture is only proven in parts and still open in its full generality. If one would be able to prove this conjecture then the differentiability assumption in conjunction with continuous group actions would turn out to be redundant.

Newmann [New31] and Smith [Smi41] have shown that a non-Lie group G acting effectively on a finite dimensional manifold necessarily contains a subgroup isomorphic to the group of p -adic integers \mathcal{Z}_p for some prime p . Thus the Hilbert–Smith conjecture can be proven if it is possible to show, that the p -adic integers cannot act effectively on any finite dimensional manifold.

In 1960 C.T. Yang showed in his article about p -adic transformation groups [Yan60] that if \mathcal{Z}_p acts effectively on an n -dimensional manifold M , then the orbit space M/\mathcal{Z}_p has homology dimension $n + 2$.¹ Yang’s paper yields a possible approach to prove the Hilbert–Smith conjecture: if one succeeds in showing that the canonical projection map $\pi: M \rightarrow M/\mathcal{Z}_p$ onto the orbit space does not raise the homology dimension by 2 then this leads to a contradiction with Yang’s result and thus proves the conjecture.

Though Yang’s result has been known for long, it was not before the late 90’s that it was used in an effective way. Repovš and Ščepin showed in 1997 using the above reasoning that the Hilbert–Smith conjecture holds for Lipschitz maps [RŠ97] and in the same year Malešič affirmed the conjecture for Hölder actions [Mal97]. As a last example for the successful application of Yang’s result may serve Martin’s article from the year 1999 [Mar99] where he proved the conjecture for quasiconformal actions.

2. Yang’s Article on p -Adic Transformation Groups

Yang’s article consists mainly of two parts. The first part is devoted to the construction of certain long exact homology sequences in a modified version of Smith homology theory. In the second part the author uses the previously acquired results to prove the main result. The proof is first carried out for manifolds and free actions of the p -adic integers \mathcal{Z}_p , that is, Yang proves first the following result.

Theorem. *Let M be an n -dimensional manifold. Assume that the group of p -adic integers \mathcal{Z}_p is acting freely on M . Then the homology dimension of the orbit space M/\mathcal{Z}_p is precisely $n + 2$.*

¹See Definition 1.51 and Definition 1.52.

After this Yang strengthens this result by proving that the same result holds for more general spaces, namely homology n -manifolds², and that it remains true for an effective action of \mathcal{Z}_p . That is, Yang proves finally the theorem in following form.

Theorem. *Let M be a homology n -manifold. Assume that the group of p -adic integers \mathcal{Z}_p is acting effectively on M . Then the homology dimension of the orbit space M/\mathcal{Z}_p is precisely $n + 2$.*

3. Aim and Outline of this Thesis

This thesis focuses only on the first part of Yang's article. An essential part of Smith homology theory are certain homology groups which Smith names "special homology groups". The aim is to explain Yang's construction of the modified Smith theory for p -adic actions and Čech homology theory for compact spaces. Shortcomings in the presentation in Yang's article are rectified and thereby the comprehensibility of the construction process is improved. Furthermore some minor mistakes in Yang's article are corrected and one result is improved, see Section 4 in Chapter 2.

My thesis consists of mainly two parts. In the first part, that is Chapter 1, we will gather the working tools needed for the construction process. They are all standard working tools from different mathematical areas in Algebra and Topology. Due to its introductory flavor the first chapter will not give proofs for the results but rather refers mostly to the literature. The main aim is to collect the results and give and form a uniform notation. A main emphasis lies on the construction of Čech homology theory for compact pairs as this is the homology theory used by Yang in his article and the construction of the Smith theory is carried out for this kind of this homology theory.

In the second part of this thesis the actual construction of Smith homology theory for p -adic transformation groups is performed. The construction is carried out in three steps. First it is done for the action of cyclic p -groups on simplicial complexes, then the definition is extended to the Čech homology groups of compact topological pairs where cyclic p -groups act on those topological pairs. And then finally the construction is extended to the action of the group of p -adic integers by using the fact that the p -adic integers form an inverse limit of cyclic p -groups.

Coverings and their nerves play the essential link between the topological properties of a space and the algebraic construction of the Čech homology groups. In order to be able to make the extension of the Smith theory developed for simplicial complexes to Čech homology theory we will need certain coverings which satisfy some specific properties. This leads to the definition of "special coverings" which are introduced and discussed in Section 2 of Chapter 2 before the extension of the Smith theory to Čech homology theory.

In every step of the construction process we will study a collection of homomorphisms which arise in a natural way and certain long exact homology sequences which involve the special homology groups. It will be studied whether it is possible to extend the definition of the special homology groups in every step of the construction process and whether the homomorphisms studied carry over to each new step. It will turn out that everything works in a nice way until we reach the last step of the construction process. Still not all constructions fail and enough remains working for deriving interesting results. In the last section of Chapter 2 we will then finally use the homology dimension to improve the obtained results.

²See [Yan58] and [Yan60] for the definition of homology n -manifolds

4. Prerequisites and Notation

In this thesis it is assumed that the reader is familiar with the basic concepts and results of point-set topology, algebraic topology and transformation groups. In particular the reader is assumed to be familiar with the concept of categories admissible for homology theory³ and the Eilenberg–Steenrod axioms for homology theories as described by Eilenberg and Steenrod in their classical book about the foundations of algebraic topology [ES52].

In particular it shall be pointed out that we will use throughout this thesis the convention that *compactness implies the Hausdorff condition* for topological spaces. That is, every compact or locally compact space is assumed to be Hausdorff.

The set of natural numbers is denoted by \mathbb{N} and 0 is considered to be an element of \mathbb{N} . The ring of integers will be denoted by \mathbb{Z} and the set of real numbers by \mathbb{R} . The ring of integers modulo n are denoted by $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$, where the symbol “:=” denotes that the left side is by definition equal to the right side. The group of the reals modulo the integers will be denoted by $\mathcal{R} := \mathbb{R}/\mathbb{Z}$. The group of p -adic integers – see Section 7 of Chapter 1 – is denoted in this thesis by \mathcal{Z}_p . The power set of a set A is denoted in symbols by $\mathcal{P}(A)$.

It shall be pointed out that in this thesis we use a naming scheme which differs in one essential way from the scheme introduced in Yang’s paper, see Remark 2.21 on page 29. This is done because I believe that this notation is more intuitive than the original one.

³In the sense of [ES52].

CHAPTER 1

Working Tools

We shall introduce in this chapter all the necessary mathematical working tools needed for this thesis.

1. Simplicial Complexes and Chain Complexes

Definition 1.1. A *simplicial complex* K is a collection of non-empty finite sets such that $s' \in K$ for every $s \in K$ and any $\emptyset \neq s' \subset s$. An element $s \in K$ is said to be a *simplex*, a non-empty subset s' of a simplex $s \in K$ is called a *face* of s . If $s \in K$ is a simplex then an element $v \in s$ is called a *vertex*.

The *dimension* of a simplex s is one less than the number of vertices of s , in symbols $\dim(s) := |s| - 1$. A simplex of dimension n is also called an n -simplex. The *dimension* of a simplicial complex K is the least integer $n \geq -1$ such that $\dim(s) \leq n$ for every $s \in K$ or infinity if no such integer exists. The dimension of a simplicial complex K is denoted by $\dim(K)$. A simplicial complex is said to be *finite* if the set K is finite. A simplicial complex K is said to be *finite dimensional* if $\dim(K) < \infty$.

A subset $L \subset K$ of a simplicial complex is a (*simplicial*) *subcomplex* if L is a simplicial complex. A *simplicial pair* (K, L) is a tuple consisting of a simplicial complex K and a subcomplex $L \subset K$.

Note that it is customary to identify the vertices of K with the 0-simplices of K , and that we will also identify the simplicial pair (K, \emptyset) with the simplicial complex K .

Let K and K' be two simplicial complexes and denote by V and V' the set of vertices of K and K' respectively. Then a map $f: V \rightarrow V'$ induces in a natural way a map $f: \mathcal{P}(V) \rightarrow \mathcal{P}(V')$. Note that K is a subset of $\mathcal{P}(V)$ and $f(K)$ is a subset of $\mathcal{P}(V')$ but not necessarily of K' .

Definition 1.2. A map $f: K \rightarrow K'$ is a *simplicial map* if there exists a map $\tilde{f}: V \rightarrow V'$ such that the restriction of the induced map $\tilde{f}: \mathcal{P}(V) \rightarrow \mathcal{P}(V')$ to the simplicial complex K coincides with f . If (K, L) and (K', L') are simplicial pairs, then a *map of simplicial pairs* is a simplicial map $f: K \rightarrow K'$ such that $f(L) \subset L'$.

Given a simplicial pair (K, L) and an R -module¹ \mathcal{G} (for example an abelian group) we can construct a chain complex $C(K, L; \mathcal{G})$ as follows.

By an *ordered k -simplex* of K we shall mean a $(k + 1)$ -tuple (v_0, \dots, v_k) such that the set $\{v_0, \dots, v_k\}$ is a simplex of K . Note that the vertices of the ordered k -simplex need not to be distinct and therefore $\{v_0, \dots, v_k\}$ can be a simplex of dimension less than k . The set of all ordered simplices of K is denoted by K_{\circ} and the set of all ordered k -simplices of K is denoted by $K_{\circ, k}$. Note that if $K \neq \emptyset$ then there exists infinitely many ordered simplices.

We denote the free abelian group generated by the set $K_{\circ, k}$ by $C_k(K)$. If (K, L) is a simplicial pair, then $L_{\circ, k}$ is a subset of $K_{\circ, k}$ and $C_k(L)$ a subgroup of $C_k(K)$.

¹Where R is a commutative ring with unit. However, even though we admit this generality in our definition we will use only \mathbb{Z} -modules in this thesis, that is we will consider only abelian coefficient groups.

We define $C_k(K, L) := C_k(K)/C_k(L)$ and it follows that $C_k(K, L)$ can be identified with the free abelian group generated by the ordered k -simplices of K which are *not* ordered k -simplices of L .

Given an ordered k -simplex (v_0, \dots, v_k) of K we define

$$\partial_k(v_0, \dots, v_k) := \sum_{i=0}^k (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_k)$$

where $(v_0, \dots, \hat{v}_i, \dots, v_k)$ denotes the ordered $(k-1)$ -simplex of K which is derived from (v_0, \dots, v_k) by leaving out the i -th vertex. It follows that this defines a group homomorphism $\partial_k: C_k(K, L) \rightarrow C_{k-1}(K, L)$. Further it follows from a standard proof that $\partial_{k-1}\partial_k = 0$. Therefore the groups $C_k(K, L)$ together with the homomorphisms ∂_k define a non-negative chain complex. The homomorphisms ∂_k are called boundary homomorphisms.

Definition 1.3 (Ordered Chain Complex). Let (K, L) be a simplicial pair. Then the *ordered chain complex* $C(K, L)$ of the pair (K, L) is the chain complex

$$C(K, L) := \{C_k(K, L), \partial_k\}.$$

If $L = \emptyset$ then we use the abbreviation $C(K) := C(K, \emptyset)$.

If \mathcal{G} is an R -module, then the tensor product $C_k(K, L; \mathcal{G}) := C_k(K, L) \otimes \mathcal{G}$ becomes an R -module in a natural way. Further it follows that (by abuse of notation) $\partial_k := \partial_k \otimes \text{id}$ makes $\{C_k(K, L; \mathcal{G}), \partial_k\}$ into a non-negative chain complex.

Definition 1.4. The chain complex

$$C(K, L; \mathcal{G}) := \{C_k(K, L; \mathcal{G}), \partial_k\}$$

is called the chain complex of the simplicial pair (K, L) *with coefficients in* \mathcal{G} . As in the definition above we use the abbreviation $C(K; \mathcal{G}) := C(K, L; \mathcal{G})$ in case $L = \emptyset$.

For obvious reasons one can identify in a natural way the chain complex $C(K, L)$ with $C(K, L; \mathbb{Z})$.

If $f: (K, L) \rightarrow (K', L')$ is a map of simplicial pairs, then f induces by

$$f(v_0, \dots, v_k) := (f(v_0), \dots, f(v_k))$$

a map $f: K_{o,k} \rightarrow K'_{o,k}$ such that $f(L_{o,k}) \subset L'_{o,k}$. This map, defined on the generators of $C_k(K)$, defines a homomorphism $f: C_k(K) \rightarrow C_k(K')$ and since $f(L_{o,k}) \subset L'_{o,k}$ this homomorphism induces a homomorphism

$$f_k: C_k(K, L) \rightarrow C_k(K', L').$$

By abuse of notation we get finally that $f_k := f_k \otimes \text{id}$ defines an R -module homomorphism $f_k: C_k(K, L; \mathcal{G}) \rightarrow C_k(K', L'; \mathcal{G})$. It follows from the definition of the boundary maps ∂_k that $f_{k-1}\partial_k = \partial_k f_k$ for every integer k . Therefore the f_k define a chain map $f: C(K, L; \mathcal{G}) \rightarrow C(K', L'; \mathcal{G})$ of degree 0.

Definition 1.5. Let (K, L) be a simplicial pair and \mathcal{G} an R -module. Then the *module of k -cycles* of $C(K, L; \mathcal{G})$ is the R -module $Z_k(K, L; \mathcal{G}) := \ker \partial_k$ and the *module of k -boundaries* of $C(K, L; \mathcal{G})$ is the R -module $B_k(K, L; \mathcal{G}) := \text{im } \partial_{k+1}$. The *k -th homology group* of $C(K, L; \mathcal{G})$ is the quotient group

$$H_k(K, L; \mathcal{G}) := Z_k(K, L; \mathcal{G})/B_k(K, L; \mathcal{G}).$$

and this group has the structure of an R -module. In the case of the coefficient group \mathcal{G} being the group of integers \mathbb{Z} we omit it in the notation for obvious reasons, that is we write then $Z_k(K, L)$ instead of $Z_k(K, L; \mathbb{Z})$, $B_k(K, L)$ instead of $B_k(K, L; \mathbb{Z})$ and $H_k(K, L)$ instead of $H_k(K, L; \mathbb{Z})$.

If $f: C(K, L; \mathcal{G}) \rightarrow C(K', L'; \mathcal{G})$ is a chain map (for example induced by a map of simplicial pairs $f: (K, L) \rightarrow (K', L')$) then this map induces in the usual way a homomorphism $f_*: H_k(K, L; \mathcal{G}) \rightarrow H_k(K', L'; \mathcal{G})$ for every integer k .

If $g: C(K, L; \mathcal{G}) \rightarrow C(K', L'; \mathcal{G})$ is another chain map, then by a *chain homotopy* $D: f \simeq g$ we mean a collection of homomorphisms

$$D_k: C_k(K, L; \mathcal{G}) \rightarrow C_{k+1}(K', L'; \mathcal{G})$$

such that $D_{k-1}\partial_k + \partial_{k+1}D_k = f_k - g_k$ for every integer k . If such a chain homotopy exists then we say that the chain maps f and g are *chain homotopic*. It follows from a standard proof that chain homotopic maps f and g induce equal maps in homology, that is $f_* = g_*$.

The natural question is, when do two maps of simplicial pairs $f, g: (K, L) \rightarrow (K', L')$ induce chain homotopic maps $f, g: C(K, L; \mathcal{G}) \rightarrow C(K', L'; \mathcal{G})$. A simple necessity for this to happen is given by the concept of contiguity of simplicial maps. We say that f and g are *contiguous* if for every simplex s of K the union $f(s) \cup g(s)$ is a simplex in K' and if for every simplex s' of L the union $f(s') \cup g(s')$ is a simplex in L' .

Lemma 1.6. *Let $f, g: (K, L) \rightarrow (K', L')$ be contiguous maps of simplicial pairs. Then $f, g: C(K, L; \mathcal{G}) \rightarrow C(K', L'; \mathcal{G})$ are chain homotopic maps. In particular $f_* = g_*$ when passing to homology.*

PROOF. One verifies that a chain homotopy $D: f \simeq g$ is given by

$$D_k(v_0, \dots, v_k) := \sum_{i=0}^k (-1)^i (f(v_1), \dots, f(v_i), g(v_i), \dots, g(v_k)). \quad (1.7)$$

A detailed proof of this result can be found in [ES52, pp. 164f.]. \square

We recall two standard results about chain complexes which will be of importance later. For a proof of these results see for example [Spa66].

Proposition 1.8. *For every short exact sequence*

$$0 \longrightarrow C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \longrightarrow 0$$

of chain complexes and chain maps of degree 0 one can define for every integer k a homomorphism $\partial_*: H_k(C'') \rightarrow H_{k-1}(C')$ by $\partial_* := (\alpha^{-1} \circ \partial \circ \beta^{-1})_*$ where the ∂ on the righthand side is the boundary homomorphism of the chain complex C . This homomorphism is also called the boundary homomorphism (or connecting homomorphism) and it has the following naturality property:

If we have another short exact sequence $0 \rightarrow \bar{C}' \rightarrow \bar{C} \rightarrow \bar{C}'' \rightarrow 0$ of chain complexes and chain maps of degree 0 and if f', f and f'' are three chain maps of degree 0 such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 & \longrightarrow & \bar{C}' & \longrightarrow & \bar{C} & \longrightarrow & \bar{C}'' & \longrightarrow & 0 \end{array}$$

commutes, then $\partial_* f'' = f'_* \partial_*$. \square

Proposition 1.9. *For every short exact sequence*

$$0 \longrightarrow C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \longrightarrow 0$$

of chain complexes and chain maps of degree 0 the long homology sequence

$$\dots \xrightarrow{\partial_*} H_k(C') \xrightarrow{\alpha_*} H_k(C) \xrightarrow{\beta_*} H_k(C'') \xrightarrow{\partial_*} H_{k-1}(C') \xrightarrow{\alpha_*} \dots$$

is exact. □

If \mathcal{G} is a compact topological group and if (K, L) is a finite simplicial pair, then we can topologize the homology groups $H_k(K, L; \mathcal{G})$ using the following construction (for details see [ES52, pp. 140ff.]).

Assume that F is a finitely generated free abelian group, say the generators of F are c_1, \dots, c_n . Then for each $c \in F \otimes \mathcal{G}$ there exist unique elements $g_1, \dots, g_n \in \mathcal{G}$ such that

$$c = \sum_{i=1}^n c_i \otimes g_i.$$

If one defines with this notation $\varphi(c) := (g_1, \dots, g_n)$ for every $c \in F \otimes \mathcal{G}$, then this yields an isomorphism

$$\varphi: F \otimes \mathcal{G} \xrightarrow{\cong} \mathcal{G}^n.$$

This isomorphism can be used to transfer the topology of \mathcal{G}^n to $F \otimes \mathcal{G}$. In this way $F \otimes \mathcal{G}$ becomes a compact group since \mathcal{G}^n is compact. One can verify that this topology is independent of the choice of the isomorphism φ .

Furthermore it follows that if F' is another finitely generated free abelian group and $f: F \rightarrow F'$ is a homomorphism, then the homomorphism

$$f \otimes \text{id}: F \otimes \mathcal{G} \rightarrow F' \otimes \mathcal{G}$$

is continuous (which is even true in a more general form, but the above result is enough for our purposes).

We can use this to topologize the homology groups $H_k(K, L; \mathcal{G})$ of a *finite* simplicial pair (K, L) . Due to the finiteness of the simplicial pair we get that the group $C_k(K, L)$ is finitely generated for every integer k . Thus by the above construction the $C_k(K, L; \mathcal{G}) = C_k(K, L) \otimes \mathcal{G}$ become compact groups. Since the boundary homomorphisms ∂_k of the chain complex $C(K, L; \mathcal{G})$ were defined to be

$$\partial_k \otimes \text{id}: C_k(K, L; \mathcal{G}) \rightarrow C_{k-1}(K, L; \mathcal{G})$$

it follows from the above considerations that they are continuous maps.

Now $B_k(K, L; \mathcal{G}) = \text{im } \partial_{k+1}$ is compact as the continuous image of a compact group. In particular, since we assumed that compactness always includes the Hausdorff condition, $B_k(K, L; \mathcal{G})$ is a closed subgroup of $C_k(K, L; \mathcal{G})$. Moreover $Z_k(K, L; \mathcal{G}) = \ker \partial_k = \partial_k^{-1}(\{0\})$ is a closed subgroup of the compact group $C_k(K, L; \mathcal{G})$ and thus itself again compact. Therefore

$$H_k(K, L; \mathcal{G}) = Z_k(K, L; \mathcal{G}) / B_k(K, L; \mathcal{G})$$

is a compact group since it is the quotient group of the compact group $Z_k(K, L; \mathcal{G})$ and the closed subgroup $B_k(K, L; \mathcal{G})$.

Note that any map of finite simplicial pairs $f: (K, L) \rightarrow (K', L')$ induces a chain map $f \otimes \text{id}: C_k(K, L; \mathcal{G}) \rightarrow C_k(K', L'; \mathcal{G})$ which is continuous. Therefore the induced maps $f_*: H_k(K, L; \mathcal{G}) \rightarrow H_k(K', L'; \mathcal{G})$ are continuous for every integer k . This statement completes the topologization of the homology groups for finite simplicial pairs (K, L) and compact coefficient groups \mathcal{G} .

2. Simplicial G -Complexes

We turn our attention to the situation where K is a simplicial complex where a finite abelian group G acts on K via simplicial maps. In this case we say that K is a *simplicial G -complex*. If (K, L) is a simplicial pair on which G acts on via maps of simplicial pairs then we say that (K, L) is a *simplicial G -pair*. The latter definition amounts to a simplicial G -complex K with a G -invariant subcomplex L .

Note that the action of G on a simplicial pair (K, L) extends in a natural way to an action of G on the set K_\circ such that L_\circ is a G -invariant subset of K_\circ . Note further that ordered k -simplices remain ordered k -simplices under the action of G .

Given a set X and a group G acting on X two objects are usually of interest: the fixed point set $X^G := \{x \in X : gx = x \text{ for all } g \in G\}$ and the orbit space $X/G := \{G(x) : x \in X\}$ (where $G(x) := \{gx : g \in G\}$ denotes the orbit of x under the action of G). Further, there exists a canonical projection $\pi: X \rightarrow X/G$ onto the orbit space which maps $x \mapsto G(x)$.

We would like to apply these concepts to a G -complex K such that K^G and K/G are again simplicial complexes and such that the canonical projection is a simplicial map. If we do this without additional assumptions on K and in a naive way then we run very quickly in several places into trouble. For instance, even though K^G is a subset of the simplicial complex K it is not necessarily a subcomplex as there might be G -invariant simplices $s \in K$ which have faces which are not left invariant under the action of G . Even worse, the orbit space K/G does not have the structure of a simplicial complex at all and neither can then the canonical projection onto the orbit space be a simplicial map.

In the following we shall discuss under which assumptions we can avoid the above mentioned problems. Following [Bre72] and [Kaw91] we will introduce two properties – property P_1 and the stronger property P_2 – for G -complexes which then ensure that K^G is a subcomplex and that we can endow K/G with a structure of a simplicial complex.

Definition 1.10. Let K be a G -complex. We say that K satisfies the *property P_1* if g leaves the set $s \cap gs$ pointwise fixed for every $g \in G$ and every simplex $s \in K$.

Note that if K satisfies the property P_1 and L is a subset of K , then L satisfies the property P_1 , too. In particular this is true if L is a subcomplex of K .

Proposition 1.11. *Let K be a G -complex satisfying property P_1 . Then the fixed point set $K^G = \{s \in K : gs = s \text{ for every } g \in G\}$ consists precisely of those simplices $s \in K$ whose vertices are pointwise fixed by the action of G . In particular K^G is a subcomplex of K . \square*

For a proof of this result – as for the proofs of the following results in this section – we shall refer to [Kaw91, pp. 227–233].

Lemma 1.12. *Let K be a G -complex satisfying property P_1 . Let V be the set of vertices of K . Then G acts on V . Denote by $\pi: V \rightarrow V/G$ the canonical projection onto the orbit space. If $\{v_0, \dots, v_k\}$ is a k -simplex of K then $\pi(v_0), \dots, \pi(v_k)$ are $k+1$ distinct elements in the orbit space V/G . \square*

Now the canonical projection $\pi: V \rightarrow V/G$ onto the orbit space extends to a map $\pi: \mathcal{P}(V) \rightarrow \mathcal{P}(V/G)$. It follows that the restriction of this map to K defines a simplicial map $\pi: K \rightarrow \pi(K)$ and by the previous lemma this simplicial map sends k -simplices of K to k -simplices of $\pi(K)$. Yet we need still additional constraints on K so that we can identify the simplicial complex $\pi(K)$ with the orbit space K/G in a useful way.

Definition 1.13. Let K be a G -complex. We say that K satisfies the *property P_2* if whenever $s = \{v_0, \dots, v_k\}$ and $s' = \{g_0v_0, \dots, g_kv_k\}$ are simplices of K (with $g_i \in G$), then there exists a $g \in G$ such that $s' = gs$.

We say that K is a *regular G -complex* if K satisfies property P_2 as an H -complex for every subgroup H of G .

Note that if K satisfies property P_2 then this is true for any subset L of K . In particular this is true for any subcomplex L of K .

Lemma 1.14. *Let K be a simplicial G -complex which satisfies property P_2 . Then K satisfies property P_1 , too.* \square

Proposition 1.15. *Let K be a regular G -complex. Then the assignment*

$$G(s) \mapsto \pi(s) \tag{*}$$

for all $s \in K$ defines a bijection $K/G \mapsto \pi(K)$ such that the diagram

$$\begin{array}{ccc} & & K/G \\ & \nearrow \pi & \downarrow \\ K & & \\ & \searrow \pi & \\ & & \pi(K) \end{array}$$

commutes. In this diagram the map $\pi: K \rightarrow K/G$ is the canonical projection onto the orbit space K/G and the map $\pi: K \rightarrow \pi(K)$ is the simplicial map induced by the canonical projection $\pi: V \rightarrow V/G$ onto the orbit space V/G . \square

Note the abuse of notation in the diagram of the above proposition: we used the same label π for two different maps. But if we use the identification of K/G and $\pi(K)$ as given by (*) then those two maps can be viewed to be one and the same and we will from now on use the symbol K/G to denote the simplicial complex $\pi(K)$. This then justifies the following definition.

Definition 1.16. Let K be a regular G -complex. Then the *simplicial orbit space* of K is

$$K/G := \pi(K).$$

3. Inverse Limits

Inverse limits will play an essential role in this thesis. Čech homology is defined by means of inverse systems of homology groups and the p -adic integers are the limit of an inverse system of p -groups. In addition to the basic definitions related to inverse limits we shall state some basic results. In this thesis inverse limits are used for compact topological spaces and topological groups only. Therefore we shall introduce inverse limits only in this setting. This section follows very closely [ES52, pp. 212ff.] and for the proofs of the results in this section we shall refer to this work.

Definition 1.17 (Directed Set). A *directed set* $M = (M, \leq)$ is a pre-ordered set (that is the relation “ \leq ” on M is reflexive and transitive) which satisfies the following condition: for all $\alpha, \beta \in M$ there exists a $\gamma \in M$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

A directed set $M' = (M', \preceq)$ is said to be a *subset* of M (in symbols $M' \subset M$) if $\alpha \in M'$ implies $\alpha \in M$ and $\alpha \preceq \beta$ implies $\alpha \leq \beta$. A subset $M' \subset M$ is said to be *cofinal* if for each $\alpha \in M$ there exists a $\beta \in M'$ such that $\alpha \leq \beta$.

If (M, \leq) and (M', \preceq) are two directed sets, then by a *map of directed sets* $\varphi: M \rightarrow M'$ we understand an order preserving map, that is $\alpha \leq \beta$ implies $\varphi(\alpha) \preceq \varphi(\beta)$ for every $\alpha, \beta \in M$.

Note that the relation “ \leq ” is not required to be antisymmetric, that is $\alpha \leq \beta$ together with $\beta \leq \alpha$ does not necessarily imply $\alpha = \beta$.

The natural numbers \mathbb{N} become a directed set with the natural ordering “ \leq ” relation. When we use the natural numbers in this thesis as a directed set we shall always assume this ordering. Another example of a directed set is the topology of a topological space ordered by inclusion.

Definition 1.18 (Inverse System). Let $M = (M, \leq)$ be a directed set. An *inverse system* $\{X, \pi\}$ is a function which assigns each $\alpha \in M$ a set X_α and each pair $\alpha \leq \beta$ in M a map

$$\pi_\alpha^\beta: X_\beta \rightarrow X_\alpha$$

such that π_α^α is the identity for all $\alpha \in M$ and

$$\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$$

for all $\alpha \leq \beta \leq \gamma$ in M . The maps π_α^β are called the *projections* of the inverse system.

Let $\{X', \pi'\}$ be another inverse system directed by a set M' . Then by a *map of inverse systems*

$$\Phi: \{X, \pi\} \rightarrow \{X', \pi'\}$$

we understand a map $\varphi: M' \rightarrow M$ together with a collection of maps

$$\varphi_{\alpha'}: X_{\varphi(\alpha')} \rightarrow X'_{\alpha'}, \quad \alpha' \in M',$$

such that for all $\alpha' \leq \beta'$ in M' the diagram

$$\begin{array}{ccc} X_{\varphi(\beta')} & \xrightarrow{\varphi_{\beta'}} & X'_{\beta'} \\ \pi_{\varphi(\alpha')}^{\varphi(\beta')} \downarrow & & \downarrow \pi'_{\alpha'}^{\beta'} \\ X_{\varphi(\alpha')} & \xrightarrow{\varphi_{\alpha'}} & X'_{\alpha'} \end{array}$$

commutes.

We are now ready to define the limit of inverse systems and maps between them

Definition 1.19 (Inverse Limit). Let $\{X, \pi\}$ be an inverse system directed by the set M . The *inverse limit* of $\{X, \pi\}$ is the subset

$$X_\infty \subset \prod_{\alpha \in M} X_\alpha$$

which consists of all those elements $\{x_\alpha\}_{\alpha \in M}$ such that $\pi_\alpha^\beta(x_\beta) = x_\alpha$ for all $\alpha \leq \beta$ in M . In symbols the inverse limit is denoted by

$$\varprojlim X_\alpha := X_\infty$$

For each $\alpha \in M$ we denote the projections to the factors X_α by $\pi_\alpha: X_\infty \rightarrow X_\alpha$.

Let $\{X', \pi'\}$ be another inverse system directed by the set M' and let

$$\Phi: \{X, \pi\} \rightarrow \{X', \pi'\}$$

be a map of inverse systems. For each $x = \{x_\alpha\}_{\alpha \in M} \in X_\infty$ define $x'_{\alpha'} := \varphi_{\alpha'}(x_{\varphi(\alpha)})$ for every $\alpha' \in M'$. Then it follows that $x' = \{x'_{\alpha'}\}_{\alpha' \in M'}$ is an element of X'_∞ and thus we get by the assignment $x \mapsto x'$ a map

$$\varphi_\infty: X_\infty \rightarrow X'_\infty$$

which is called the *inverse limit* of the map Φ , in symbols

$$\varprojlim \Phi := \varphi_\infty$$

Note that the above definitions yield the commutative diagram

$$\begin{array}{ccccc}
 X_{\varphi(\beta')} & \xrightarrow{\varphi} & X'_{\beta'} & & \\
 \downarrow \pi & \swarrow \pi_{\varphi(\beta')} & \nearrow \pi_{\beta'} & & \downarrow \pi' \\
 & X_\infty & \xrightarrow{\varphi_\infty} & X'_\infty & \\
 & \swarrow \pi_{\varphi(\alpha')} & \searrow \pi_{\alpha'} & & \\
 X_{\varphi(\alpha')} & \xrightarrow{\varphi} & X'_{\alpha'} & &
 \end{array}$$

for all $\alpha' \preceq \beta'$ in M' . Note further that the existence of the map $\Phi: \{X, \pi\} \rightarrow \{X', \pi'\}$ ensures that X'_∞ is not the empty set in case that X_∞ is not the empty set.

The inverse systems encountered in this thesis will consist of topological spaces, groups or topological groups. The projections of those inverse systems will be continuous maps, homomorphism and continuous homomorphisms respectively. Maps between inverse systems of topological spaces, groups or topological groups will consist of maps of belonging to the same category respectively. It follows that then also the limits of the inverse systems and the maps between them belong to the same category.

The limit of an inverse system of groups and group homomorphisms always contains the neutral element and therefore the inverse limit is non-empty, too. The limit X_∞ of non-empty compact spaces and continuous projections $\{X_\alpha, \pi\}$ can be shown to be a non-empty closed subspace of $\prod X_\alpha$ and therefore we get the following result.

Proposition 1.20. *Let $\{X, \pi\}$ be an inverse system of non-empty compact spaces and continuous projections. Then the limit*

$$X_\infty = \varprojlim X_\alpha$$

is a compact space and not empty. In particular the inverse limit of compact groups is compact. \square

And we shall add one more technical result about the inverse limit which we will need at the end of this thesis.

Proposition 1.21. *Let $\{X, \pi\}$ be an inverse system of compact spaces directed by M and $\alpha \in M$. If U is an open subset of X_α such that $\pi_\alpha(X_\infty) \subset U$, then there exists a $\beta \geq \alpha$ such that $\pi_\alpha^\beta(X_\beta) \subset U$.* \square

Definition 1.22. Let $\{X, \pi\}$ be an inverse system directed by the set M and let M' be a subset of M as a directed set. If $\{X', \pi'\}$ is the restriction of $\{X, \pi\}$ to the directed set M' then it is called a *subsystem* of $\{X, \pi\}$. If M' is a cofinal subset of M then $\{X', \pi'\}$ is called a *cofinal* subsystem of $\{X, \pi\}$.

If $\{X', \pi'\}$ is a subsystem of $\{X, \pi\}$ then the inclusion $\varphi: M' \rightarrow M$ together with the identities $\varphi_{\alpha'}: X_{\varphi(\alpha')} \rightarrow X'_{\alpha'}$, form a map $\Phi: \{X, \pi\} \rightarrow \{X', \pi'\}$ of inverse systems and is called the *injection of the system into the subsystem*.

Note the counter-intuitive notation in the above definition: when speaking of inverse systems an injection is indeed going from the larger system $\{X, \pi\}$ into the smaller subsystem $\{X', \pi'\}$! The importance of cofinal subsystems becomes visible with the following result.

Proposition 1.23. *Let $\{X, \pi\}$ be an inverse system and assume that $\{X', \pi'\}$ is a cofinal subsystem. Then the inverse limit of the injection $\Phi: \{X, \pi\} \rightarrow \{X', \pi'\}$ is a bijection. \square*

Corollary 1.24. *Let M' be a cofinal subset of M and let $\{X, \pi\}$ be an inverse system directed by M . If x and y are two elements of its limit X_∞ then $x = y$ if and only if $x_\alpha = y_\alpha$ for every $\alpha \in M'$. \square*

We still need to define what we mean by inverse systems of descending sequences of groups and their limits. Therefore recall the following notation: a *descending series of groups* $S = \{G_k, f_k\}$ is a collection of groups G_k indexed by the integers together with a collection of group homomorphisms $f_k: G_k \rightarrow G_{k-1}$.

$$S : \dots \longrightarrow G_k \xrightarrow{f_k} G_{k-1} \longrightarrow \dots \longrightarrow G_1 \xrightarrow{f_1} G_0 \xrightarrow{f_0} G_{-1} \longrightarrow \dots$$

The series is said to be of *order 2* if $f_{k-1}f_k = 0$ (that is $\text{im } f_k \subset \ker f_{k-1}$) for all integers k . The series is said to be *exact* if $\text{im } f_k = \ker f_{k-1}$ for all integers k .

If $S' = \{G'_k, f'_k\}$ is another descending series of groups, then a *map (of degree 0)* $\Psi: S \rightarrow S'$ is a collection of group homomorphisms $\psi_k: G_k \rightarrow G'_k$ such that $f'_k\psi_k = \psi_{k-1}f_k$ for all integers k .

Again, if we speak of descending series of topological groups and maps between them then all the homomorphisms involved are assumed to be continuous.

The extension of inverse systems and their limits to inverse systems of descending series of groups and their limits is done in a natural way.

Definition 1.25. By a *inverse system of descending sequences of groups* $\{S, \pi\}$ directed by a set M we understand a function which assigns each $\alpha \in M$ a descending sequence of groups $S_\alpha = \{G_\alpha, f_\alpha\}$ and to each pair $\alpha \leq \beta$ in M a map $\pi_\alpha^\beta: S_\beta \rightarrow S_\alpha$ of degree 0 such that $\{G_{\alpha,k}, \pi_{\alpha,k}^\beta\}$ forms an inverse system of groups for each integer k and such that

$$\pi_{\alpha,k-1}^\beta f_{\beta,k} = f_{\alpha,k} \pi_{\alpha,k}^\beta$$

for all $\alpha \leq \beta$ in M and all integers k .

We can then form for each integer k the inverse system $\{G_k, \pi_k\}$ directed by M . Moreover the collection $\{f_{\alpha,k}\}_{\alpha \in M}$ forms a map $F_k: \{G_k, \pi_k\} \rightarrow \{G_{k-1}, \pi_{k-1}\}$. Denote by $G_{k,\infty}$ and $f_{k,\infty}$ the respective limits. Then by the *limit S_∞ of the inverse system of descending sequences of groups* we shall understand the descending sequence of groups $\{G_{k,\infty}, f_{k,\infty}\}$, in symbols

$$\varprojlim S_\alpha := S_\infty$$

If $\{S, \pi\}$ is an inverse system of descending sequences of topological groups and continuous homomorphisms, then the limit sequence is again a descending sequence of topological groups and continuous homomorphisms. If all the groups are compact then the groups of the limit sequence are compact, too. It is also easily seen that if all the descending sequences of the inverse system are of order of 2, then also the limit sequence is of order 2.

But in general it is not true that if all the sequences of an inverse system of descending sequences are exact then also the limit sequence is exact. But in the case of sequences of compact topological groups we have the following result.

Proposition 1.26. *Let $\{S, \pi\}$ be an inverse system of descending sequences of compact topological groups directed by M . If for each $\alpha \in M$ the descending series S_α is exact then the limit sequence S_∞ is exact, too. \square*

4. Coverings of Spaces

In this section X will be a topological space and A a subset of X . In this thesis we shall mean by a (*self-indexed*) *covering* λ of A in X a collection of subsets of X such that the interior $\text{Int } U$ of every $U \in \lambda$ has a non-empty intersection with A and

$$A \subset \bigcup_{U \in \lambda} \text{Int } U.$$

In the special case that $A = X$ we say that λ is a *covering of X (in itself)*. An *open (closed) covering* is a covering which consists only of open (closed) sets. If the covering λ consists only of finite many subsets of X then λ is said to be *finite*.

Self-indexed coverings will be enough for most of the arguments in this thesis. However we need a definition which admits some more subtle constructions.

Definition 1.27 (Indexed Coverings). By an *indexed covering* α of A in X we shall mean a function $\alpha: V_\alpha \rightarrow \mathcal{P}(X)$ which assigns to each element v of an abstract index set V_α a subset α_v of X such that $\{\alpha_v : v \in V_\alpha\}$ is a covering of A in X . In the special case that $A = X$ we speak just of an *indexed covering of X (in itself)*. An indexed covering is said to be *open (closed)* if the set α_v is open (closed) for each $v \in V_\alpha$. The covering α is said to be *finite* if the index set V_α is finite

Note that any self-indexed covering λ can be identified in a natural way with the indexed covering $\alpha: V_\alpha \rightarrow \mathcal{P}(X)$ with $V_\alpha := \lambda$ and which maps each $U \in \lambda$ precisely to $U \in \mathcal{P}(X)$. On the other hand, if α is an indexed covering with index set V_α such that the map α is injective, then α can be identified in a natural way with the self-indexed covering $\lambda := \{\alpha_v : v \in V_\alpha\}$.

Note further that the only indexed covering which covers the empty set in X is $\alpha_0: \emptyset \rightarrow \mathcal{P}(X)$. This indexed covering corresponds to the empty self-indexed covering $\lambda_0 := \emptyset$, which is the only self-indexed covering which covers the empty set in X .

Definition 1.28. Let $\beta: V_\beta \rightarrow \mathcal{P}(X)$ be another covering of A in X . A *refinement projection* $p: \beta \rightarrow \alpha$ from β to α is a map $p: V_\beta \rightarrow V_\alpha$ such that $\beta_w \subset \alpha_{p(w)}$. If there exists a refinement projection $p: \beta \rightarrow \alpha$ then we say that β *refines* α .

Let $U \subset X$ be an arbitrary subset and set $S := \{v \in V_\alpha : \alpha_v \cap U \neq \emptyset\}$. The *star of U in α* is the union of all sets α_v for $v \in S$, that is

$$\text{star}_\alpha(U) := \bigcup_{v \in S} \alpha_v.$$

Definition 1.29. A *star-refinement projection* $p: \beta \rightarrow \alpha$ is a map $p: V_\beta \rightarrow V_\alpha$ such that $\text{star}_\beta(\beta_w) \subset \alpha_{p(w)}$ for every $w \in V_\beta$. If there exists a star-refinement projection $p: \beta \rightarrow \alpha$ then we say that β *star-refines* α .

We shall denote the class of all indexed open coverings of A in X by $\text{Cov}(A; X)$, and the class of all open coverings of X will be denoted by $\text{Cov}(X)$. Note that both $\text{Cov}(A; X)$ and $\text{Cov}(X)$ are proper classes and not sets since there is no restriction to the index sets. Yet we shall treat those classes in the following in a sloppy way as sets. In [ES52, p. 238] it is shown what restriction one needs to apply to those classes so that they become small enough to be sets without changing the results in an essential way.²

Note that the relation “ α is refined by β ” (in symbols we shall denote the fact by $\alpha \leq \beta$) makes the sets $\text{Cov}(A; X)$ and $\text{Cov}(X)$ into directed sets, see [ES52, p. 235].

²The rough idea is that consider only open coverings which have index sets whose cardinality does not exceed $\omega(X)$ where $\omega(X)$ is defined to be the least cardinal such that X has a base for its open sets of cardinal power $\omega(X)$.

The next result is a consequence of the definition of compactness.

Lemma 1.30. *Let X be a compact space and $A \subset X$ a closed subset. Then any open covering α of A in X can be refined by a finite open covering β of A in X . \square*

In other words, if X is a compact space and $A \subset X$ a closed subset then the set $\text{Cov}_f(A; X)$ of all finite open coverings of A in X is cofinal in the set of all open coverings of A in X .

From a standart result about paracompact spaces follows the next result.

Lemma 1.31. *Let X be a compact space and $A \subset X$ a closed set. Then any open covering of A in X can be star-refined by an open covering β . \square*

Therefore also the relation “ α is star-refined by β ” makes $\text{Cov}(A; X)$ and $\text{Cov}(X)$ into a directed set in the case of X being a compact space and A a closed subset of X .

Definition 1.32. Let $\alpha: V_\alpha \rightarrow \mathcal{P}(X)$ be an indexed collection of subsets of X . Then for $s \subset V_\alpha$ we denote by

$$\text{Car}_\alpha(s) := \bigcap_{v \in s} \alpha_v$$

the carrier of s (with respect to α). Let A be a subset of X . Then the nerve of α with respect to A is the set

$$A_\alpha := \{s \subset V_\alpha : \text{Car}_\alpha(s) \cap A \neq \emptyset \text{ and } s \text{ is finite}\}.$$

It follows immediately that the nerve A_α of α with respect to A has the structure of a simplicial complex.

Now consider the following case: Let X be a topological space and $A, B, Y \subset X$ three subsets with the only constraint that (Y, B) forms a pair of topological spaces. If α is a covering of A in X it follows that (Y_α, B_α) is a simplicial pair. If β is another covering of A in X , then (Y_β, B_β) is another simplicial pair.

Lemma 1.33. *In the above case any refinement projection $p: \beta \rightarrow \alpha$ defines a map $p: (Y_\beta, B_\beta) \rightarrow (Y_\alpha, B_\alpha)$ of simplicial pairs and this map is unique up to contiguity. \square*

In particular this means that any refinement projection $p: \beta \rightarrow \alpha$ defines a unique homomorphism $p_*: H_k(Y_\beta, B_\beta; \mathcal{G}) \rightarrow H_k(Y_\alpha, B_\alpha; \mathcal{G})$ when passing to homology.

5. The Construction of the Čech Homology Theory

In this section we shall introduce the Čech homology theory for the admissible category of compact pairs and continuous maps between them. The description of the construction is based on Chapter IX in [ES52] and for the proofs of the results presented in this section we shall refer to this work. However, we will not describe the construction process in the generality presented in this book but rather restrict ourself to closed topological pairs (X, A) . This way we avoid technical difficulties and bring the essential part of the construction process better into light. See herefore also [ES52, pp. 249f.]. This restriction is not essential to us since after all we will need the Čech homology theory only for compact and therefore closed pairs.

Throughout this section we assume that (X, A) is a closed topological pair, that is, a topological space X with a closed subset $A \subset X$. Further \mathcal{G} will denote in this section a fixed R -module with R being a commutative ring with unit. But even though we construct the Čech homology theory for this general setting we will use it later only in the case of compact pairs (X, A) and compact abelian groups \mathcal{G} as coefficients.

Recall that a homology theory for a category of topological pairs admissible for homology theory consists of three type of objects: homology groups, induced maps between homology groups and natural boundary homomorphisms. We will outline the construction process of all these objects for the Čech homology theory.

Let $\alpha, \beta \in \text{Cov}(X)$ and assume that α is refined by β . It follows by Lemma 1.33 that any refinement projection $p_\alpha^\beta: \beta \rightarrow \alpha$ induces a unique homomorphism

$$p_{\alpha*}^\beta: H_k(X_\beta, A_\beta; \mathcal{G}) \rightarrow H_k(X_\alpha, A_\alpha; \mathcal{G})$$

in homology independent of the choice of $p_{\alpha*}^\beta$.

Since $\text{id}: \alpha \rightarrow \alpha$ is always a refinement projection for any $\alpha \in \text{Cov}(X)$ it follows that $p_{\alpha*}^\alpha: H_k(X_\alpha, A_\alpha; \mathcal{G}) \rightarrow H_k(X_\alpha, A_\alpha; \mathcal{G})$ is the identity, too. Further, if $\alpha, \beta, \gamma \in \text{Cov}(X)$ are such that α is refined by β and β is refined by γ , then α is also refined by γ . If $p_\alpha^\beta: \beta \rightarrow \alpha$, $p_\beta^\gamma: \gamma \rightarrow \beta$ and $p_\alpha^\gamma: \gamma \rightarrow \alpha$ are refinement projections, then apparently $p_\alpha^\beta p_\beta^\gamma: \gamma \rightarrow \alpha$ is a refinement projection, too, and thus by Lemma 1.33 we have the equality $p_{\alpha*}^\gamma = (p_\alpha^\beta p_\beta^\gamma)_*$ when passing to homology. Then since $(p_\alpha^\beta p_\beta^\gamma)_* = p_{\alpha*}^\beta p_{\beta*}^\gamma$ we get altogether the equality

$$p_{\alpha*}^\gamma = p_{\alpha*}^\beta p_{\beta*}^\gamma.$$

Putting all these pieces of information together this means in other words that

$$\{H_k(X_\alpha, A_\alpha; \mathcal{G}), p_*\} \quad (*)$$

is an inverse system of R -modules directed by $\text{Cov}(X)$.

Definition 1.34. Let (X, A) be a closed topological pair and \mathcal{G} an R -module. Then the k -th Čech homology group $\check{H}_k(X, A; \mathcal{G})$ of X modulo A with coefficients in \mathcal{G} is the limit of the inverse system $(*)$ as described above, in symbols

$$\check{H}_k(X, A; \mathcal{G}) := \varprojlim H_k(X_\alpha, A_\alpha; \mathcal{G}).$$

In the usual way we may omit A or \mathcal{G} in the notation if $A = \emptyset$ or $\mathcal{G} = \mathbb{Z}$.

Note that if \mathcal{G} is an R -module then the groups $\check{H}_k(X, A; \mathcal{G})$ have the structure of an R -module, too. But in general it is not true that $\check{H}_k(X, A; \mathcal{G})$ is a compact group for compact coefficient groups \mathcal{G} .

A sufficient condition for the latter to become true is to consider the Čech homology groups for a compact topological pair (X, A) . In this case we know by Lemma 1.30 that the set $\text{Cov}_f(X)$ of all finite coverings of X is a cofinal subset of $\text{Cov}(X)$. For every covering $\alpha \in \text{Cov}_f(X)$ we have that the simplicial complex X_α is finite and thus by the methods described towards the end of Section 1 the groups $H_k(X_\alpha, A_\alpha; \mathcal{G})$ can be topologized and become this way compact groups. If $\beta \in \text{Cov}(X)$ is a covering which is refining α , then any refinement projection $p_\alpha^\beta: \beta \rightarrow \alpha$ induces a continuous homomorphism $p_{\alpha*}^\beta: H_k(X_\beta, A_\beta; \mathcal{G}) \rightarrow H_k(X_\alpha, A_\alpha; \mathcal{G})$. Thus the collection

$$\{H_k(X_\alpha, A_\alpha; \mathcal{G}), p_*\} \quad (**)$$

is an inverse system of compact groups and continuous projections directed by $\text{Cov}_f(X)$. Thus by Proposition 1.20 it follows that its limit will be a compact group. Now the limit of the injection of the inverse system $(*)$ into the subsystem $(**)$ is an isomorphism due to Proposition 1.23 and we use this isomorphism to topologize $\check{H}_k(X, A; \mathcal{G})$ by demanding that this isomorphism is also a homeomorphism. As a result of this construction we obtain the following

Proposition 1.35. *Let (X, A) be a compact pair and let \mathcal{G} be a compact group. Then the Čech homology groups $\check{H}(X, A; \mathcal{G})$ are compact groups. \square*

If (Y, B) is another topological pair and $f: (X, A) \rightarrow (Y, B)$ a continuous map of pairs, then we need to construct the homomorphism $f_*: \check{H}_k(X, A; \mathcal{G}) \rightarrow \check{H}_k(Y, B; \mathcal{G})$ induced by f . We begin this construction by defining a map of inverse systems

$$F_*: \{H_k(X_\alpha, A_\alpha; \mathcal{G}), p_*\} \rightarrow \{H_k(Y_\alpha, B_\alpha; \mathcal{G}), p_*\}.$$

If $\alpha \in \text{Cov}(Y)$ is a covering of Y with index set V_α , then for every $v \in V_\alpha$ the preimage $f^{-1}(\alpha_v)$ is an open subset of X since f is continuous. Thus we can define a covering α' of X with the same index set V_α which assigns to each $v \in V_\alpha$ the set $\alpha'_v := f^{-1}(\alpha_v)$. Then α' is an open covering of X , that is $\alpha' \in \text{Cov}(X)$. Note that its nerve $X_{\alpha'}$ is a subcomplex of Y_α and the nerve $A_{\alpha'}$ is a subcomplex of B_α . We shall denote by f_α the inclusion $f_\alpha: (X_{\alpha'}, A_{\alpha'}) \hookrightarrow (Y_\alpha, B_\alpha)$.

Assume that $\beta \in \text{Cov}(Y)$ is an open covering of Y which is refining α . Denote by β' the covering of X which is constructed in the same way from β as we constructed α' from α . Then any refinement projection $p_\alpha^\beta: \beta \rightarrow \alpha$ induces a refinement projection $p_{\alpha'}^{\beta'}: \beta' \rightarrow \alpha'$. By construction we have the equality $f_\alpha p_{\alpha'}^{\beta'} = p_\alpha^\beta f_\beta$. When passing to homology the homomorphisms induced by the projections $p_{\alpha'}^{\beta'}$ and p_α^β do not depend on the choice of the refinement projection p_α^β and we get the commutative diagram

$$\begin{array}{ccc} H_k(X_{\beta'}, A_{\beta'}; \mathcal{G}) & \xrightarrow{f_{\beta*}} & H_k(Y_\beta, B_\beta; \mathcal{G}) \\ p_{\alpha'}^{\beta'} \downarrow & & \downarrow p_{\alpha*}^\beta \\ H_k(X_{\alpha'}, A_{\alpha'}; \mathcal{G}) & \xrightarrow{f_{\alpha*}} & H_k(Y_\alpha, B_\alpha; \mathcal{G}) \end{array}$$

which depends only on the covering $\alpha, \beta \in \text{Cov}(Y)$.

Thus the collection of so defined maps $f_{\alpha*}$, $\alpha \in \text{Cov}(X)$, defines a map $F_*: \{H_k(X_\alpha, A_\alpha; \mathcal{G}), p_*\} \rightarrow \{H_k(Y_\alpha, B_\alpha; \mathcal{G}), p_*\}$ of inverse systems and we denote its limit by $f_* := \varprojlim F_*$.

Definition 1.36. The homomorphisms

$$f_*: \check{H}_k(X, A; \mathcal{G}) \rightarrow \check{H}_k(Y, B; \mathcal{G})$$

as constructed above are called the *homomorphism induced by the map f* .

It follows that if \mathcal{G} is an R -module, then induced homomorphisms are R -module homomorphisms. If both (X, A) and (Y, B) are compact pairs and if \mathcal{G} is a compact group, then induced homomorphisms are seen to be continuous, too.

The following facts are easily verified to be true: If $f: (X, A) \rightarrow (X, A)$ is the identity then so is the induced homomorphism $f_*: \check{H}_k(X, A; \mathcal{G}) \rightarrow \check{H}_k(X, A; \mathcal{G})$. If $f: (X, A) \rightarrow (Y, B)$ and $g: (Y, B) \rightarrow (Z, C)$ are two continuous maps of closed pairs, then $g_* f_* = (fg)_*$. Thus, if \mathcal{G} is an R -module, then \check{H}_k which assigns to each closed pair (X, A) of an admissible category of topological spaces the k -th Čech homology group $\check{H}_k(X, A; \mathcal{G})$ and which assigns to each admissible map f the induced map f_* is a covariant functor to the category of R -modules. If \mathcal{G} is a compact group and each pair (X, A) of the admissible category is compact, then \check{H}_k is a covariant functor to the category of compact groups.

It remains to construct the boundary homomorphism

$$\partial_*: \check{H}_k(X, A; \mathcal{G}) \rightarrow \check{H}_{k-1}(A; \mathcal{G})$$

for the Čech homology theory. We will construct the homomorphism in two parts. First we construct a group $H_k(A; \mathcal{G})_X$ which will be in a natural way isomorphic to $\check{H}(A; \mathcal{G})$. In order to obtain the boundary homomorphism for the Čech homology theory we will combine this isomorphism together with a homomorphism

$\partial'_*: \check{H}_k(X, A; \mathcal{G}) \rightarrow H_{k-1}(A; \mathcal{G})_X$ which will be constructed from the boundary homomorphisms $\partial_*: H_k(X_\alpha, A_\alpha; \mathcal{G}) \rightarrow H_{k-1}(A_\alpha; \mathcal{G})$ for $\alpha \in \text{Cov}(X)$.

If $\alpha, \beta \in \text{Cov}(X)$ are two coverings of X such that α is refined by β , then any refinement projection $p_\alpha^\beta: \beta \rightarrow \alpha$ induces an unique homomorphism $p_{\alpha*}^\beta: H_k(A_\beta; \mathcal{G}) \rightarrow H_k(A_\alpha; \mathcal{G})$ when passing to homology. It follows that

$$\{H_k(A_\alpha; \mathcal{G}); p_*\}$$

forms an inverse system of R -modules directed by $\text{Cov}(X)$. The limit of this system will be denoted by $H_k(A; \mathcal{G})_X$ where the index X shall emphasize that we use $\text{Cov}(X)$ as the set directing the inverse systems and not $\text{Cov}(A)$ as in the definition of the Čech homology groups.

There is a natural way to construct a homomorphism $\check{H}_k(A; \mathcal{G}) \rightarrow H_k(A; \mathcal{G})_X$ which then will turn out to be an isomorphism. If $\alpha \in \text{Cov}(X)$ is a covering indexed by the set V_α then denote by V_α^A the set of all indices $v \in V_\alpha$ such that $\alpha_v \cap A \neq \emptyset$. Then

$$\alpha': V_\alpha^A \rightarrow \mathcal{P}(A), v \mapsto \alpha' := \alpha_v \cap A$$

defines an open covering of A , that is $\alpha' \in \text{Cov}(A)$. We can now define an order preserving map

$$\varphi: \text{Cov}(X) \rightarrow \text{Cov}(A)$$

by $\varphi(\alpha) := \alpha'$. Straight from the definition it follows that the simplicial complexes A_α and $A_{\varphi(\alpha)}$ are identical and we get that the identities $H_k(A_{\varphi(\alpha)}; \mathcal{G}) \rightarrow H_k(A_\alpha; \mathcal{G})$ together with the map φ form a map of inverse systems. We denote its limit by φ_∞ and it is an isomorphism

$$\varphi_\infty: \check{H}_k(A; \mathcal{G}) \rightarrow H_k(A; \mathcal{G})_X. \quad (\dagger)$$

Now given two coverings $\alpha, \beta \in \text{Cov}(X)$ with β refining α , any refinement projection $p_\alpha^\beta: \beta \rightarrow \alpha$ yields the following commutative diagram of chain complexes with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(A_\beta; \mathcal{G}) & \longrightarrow & C(X_\beta; \mathcal{G}) & \longrightarrow & C(X_\beta, A_\beta; \mathcal{G}) & \longrightarrow & 0 \\ & & \downarrow p_\alpha^\beta & & \downarrow p_\alpha^\beta & & \downarrow p_\alpha^\beta & & \\ 0 & \longrightarrow & C(A_\alpha; \mathcal{G}) & \longrightarrow & C(X_\alpha; \mathcal{G}) & \longrightarrow & C(X_\alpha, A_\alpha; \mathcal{G}) & \longrightarrow & 0 \end{array}$$

Then by Proposition 1.8 there exists a commutative diagram

$$\begin{array}{ccc} H_k(X_\beta, A_\beta; \mathcal{G}) & \xrightarrow{\partial_\beta} & H_{k-1}(A_\beta; \mathcal{G}) \\ p_{\alpha*}^\beta \downarrow & & \downarrow p_{\alpha*}^\beta \\ H_k(X_\alpha, A_\alpha; \mathcal{G}) & \xrightarrow{\partial_\alpha} & H_{k-1}(A_\alpha; \mathcal{G}) \end{array}$$

Thus the collection $\{\partial_\alpha\}$, $\alpha \in \text{Cov}(X)$ defines a map of inverse systems and its limit yields a homomorphism

$$\partial': \check{H}_k(X, A; \mathcal{G}) \rightarrow H_{k-1}(A; \mathcal{G})_X$$

for every integer k .

Definition 1.37. For a given closed pair (X, A) and integer k the *boundary homomorphism*

$$\partial_*: \check{H}_k(X, A; \mathcal{G}) \rightarrow \check{H}_{k-1}(A; \mathcal{G})$$

is the composite of the homomorphism $\partial': \check{H}_k(X, A; \mathcal{G}) \rightarrow H_{k-1}(A; \mathcal{G})_X$ with the isomorphism $\varphi_\infty^{-1}: H_{k-1}(A; \mathcal{G})_X \rightarrow \check{H}_{k-1}(A; \mathcal{G})$ where φ_∞ is the isomorphism defined in (†).

Now that we have constructed all the parts necessary for a homology theory, one can show that our construction yields indeed a homology theory for an admissible category of closed topological pairs. The proof of the following theorem is far from being trivial. It can for example be found in [ES52, pp. 236–250].

Theorem 1.38. *Let \mathcal{A} be a category of closed topological pairs admissible for homology theory and let \mathcal{G} be a R -module. Let \check{H} , f_* and ∂_* be as defined above. Then the Čech homology theory $\check{H} := (\check{H}, f_*, \partial)$ with coefficients in \mathcal{G} is a homology theory on \mathcal{A} which satisfies all the Eilenberg–Steenrod axioms for homology except for the exactness axiom. \square*

A homology theory H which satisfies all Eilenberg–Steenrod axioms except for the exactness axiom is called a *partially exact homology theory*. If we want to stress the fact that H satisfies all Eilenberg–Steenrod axioms then we may express this by saying that H is an *exact homology theory*.

Under some constraints the Čech homology theory can be shown to be even an exact homology theory. One of those constraints – and this will be the one we need in this thesis – is that the admissible category used consists of compact pairs and that at the same time the coefficient group \mathcal{G} is a compact group.

Under this condition the inclusions $i: A \hookrightarrow X$ and $j: X \hookrightarrow (X, A)$ yield a short exact sequence of chain complexes

$$0 \longrightarrow C(A_\alpha; \mathcal{G}) \xrightarrow{i} C(X_\alpha; \mathcal{G}) \xrightarrow{j} C(X_\alpha, A_\alpha; \mathcal{G}) \longrightarrow 0$$

for every $\alpha \in \text{Cov}(X)$. From Proposition 1.9 it follows that this sequence yields the long exact homology sequence

$$\dots \xrightarrow{\partial_*} H_k(A_\alpha; \mathcal{G}) \xrightarrow{i_*} H_k(X_\alpha; \mathcal{G}) \xrightarrow{j_*} H_k(X_\alpha, A_\alpha; \mathcal{G}) \xrightarrow{\partial_*} H_{k-1}(A_\alpha; \mathcal{G}) \xrightarrow{i_*} \dots$$

where all the homology groups are compact and all the homomorphisms are continuous. Together with the homomorphisms induced by refinement projections this gives an inverse system of descending sequences of compact groups directed by $\text{Cov}(X)$. Its limit is the sequence

$$\dots \xrightarrow{\partial'} H_k(A; \mathcal{G})_X \xrightarrow{i_*} \check{H}_k(X; \mathcal{G}) \xrightarrow{j_*} \check{H}_k(X, A; \mathcal{G}) \xrightarrow{\partial'} H_{k-1}(A; \mathcal{G})_X \xrightarrow{i_*} \dots$$

and this sequence is exact due to Proposition 1.26. It follows from the naturality of the boundary homomorphisms involved in this sequence that this sequence is isomorphic to the sequence

$$\dots \xrightarrow{\partial_*} \check{H}_k(A; \mathcal{G}) \xrightarrow{i_*} \check{H}_k(X; \mathcal{G}) \xrightarrow{j_*} \check{H}_k(X, A; \mathcal{G}) \xrightarrow{\partial_*} \check{H}_{k-1}(A; \mathcal{G}) \xrightarrow{i_*} \dots$$

This discussion did outline very roughly the proof of the following theorem. A more detailed discussion of this theorem can be found in [ES52, pp. 245–248].

Theorem 1.39. *The Čech homology theory \check{H} with compact coefficients \mathcal{G} is an exact homology theory on the admissible category of compact pairs and continuous maps between them. \square*

6. Special Properties of the Čech Homology Theory

In this section we will discuss some special properties of the Čech homology theory for compact spaces, namely the so called continuity property and a stronger form of the excision axiom which is satisfied by the Čech homology theory.

Let \mathcal{A} be a category admissible for a homology theory and assume that \mathcal{A} satisfies the following property: whenever $\{(X_\alpha, A_\alpha), \pi\}$ is an inverse system in \mathcal{A} , then its limit (X_∞, A_∞) is again in \mathcal{A} . An example for such a category satisfying this condition is the category \mathcal{A}_C .

If H is a partially exact homology theory on \mathcal{A} and $\{(X_\alpha, A_\alpha), \pi\}$ is an inverse system of pairs in \mathcal{A} directed by some set M , then the groups $H_k(X_\alpha, A_\alpha)$ together with the induced homomorphisms $\pi_{\alpha\beta}^k: H_k(X_\beta, A_\beta) \rightarrow H_k(X_\alpha, A_\alpha)$ form an inverse system of homology groups

$$\{H_k(X_\alpha, A_\alpha), \pi_*\}$$

directed by the set M . Further the projections $\pi_\alpha: (X_\infty, A_\infty) \rightarrow (X_\alpha, A_\alpha)$ induce homomorphisms $\pi_{\alpha*}: H_k(X_\infty, A_\infty) \rightarrow H_k(X_\alpha, A_\alpha)$ which then form a map of inverse systems

$$L: H_k(X_\infty, A_\infty) \rightarrow \{H_k(X_\alpha, A_\alpha), \pi_*\}.$$

Note that the domain of L is the trivial inverse system consisting of the single homology group $H_k(X_\infty, A_\infty)$ alone. The limit ℓ_∞ of L is then a homomorphism

$$\ell_\infty: H_k(X_\infty, A_\infty) \rightarrow \varprojlim H_k(X_\alpha, A_\alpha).$$

Definition 1.40. A partially exact homology theory H on \mathcal{A} is said to be *continuous* if for every inverse system $\{(X_\alpha, A_\alpha), \pi\}$ in \mathcal{A} the homomorphism ℓ_∞ is an isomorphism.

Loosely speaking this definition means that for a continuous homology theory we have that the homology functor H_k commutes with the inverse limit functor \varprojlim , that is

$$H_k \varprojlim (X_\alpha, A_\alpha) \cong \varprojlim H_k(X_\alpha, A_\alpha).$$

For the Čech homology theory one can show – see for example [ES52, pp. 261ff.] for the proof – that this homology theory is continuous under certain conditions. Precisely this is the following result.

Theorem 1.41. *Let \mathcal{G} be an R -module or a compact group. Then the Čech homology theory with coefficients in \mathcal{G} is a continuous homology theory on the category of compact pairs \mathcal{A}_C . \square*

Another special property of Čech homology theory is that it satisfies a much stronger form of excision. For that we first need two more definitions.

Definition 1.42. A map of topological pairs $f: (X, A) \rightarrow (Y, B)$ is said to be a *relative homeomorphism* if f maps $X \setminus A$ homeomorphically onto $Y \setminus B$.

For example the inclusion which appears in the excision axiom for homology theory is always a relative homeomorphism. But not every admissible relative homeomorphism satisfies the properties required by the excision axiom.

Definition 1.43. Let \mathcal{A} be a category admissible for homology theory and let H be a partially exact homology theory on \mathcal{A} . Then H is said to be *invariant under relative homeomorphisms*, if every admissible relative homeomorphism $f: (X, A) \rightarrow (Y, B)$ induces isomorphisms $f_*: H_k(X, A) \rightarrow H_k(Y, B)$ for every integer k .

Using the continuity of Čech homology theory on the category \mathcal{A}_C one can show – as for example is done in [ES52, pp. 266f.] – the following result.

Theorem 1.44. *The Čech homology theory on the category \mathcal{A}_C of compact pairs is invariant under relative homeomorphisms. \square*

7. The Groups of p -Adic Integers

Recall that we denote by $[r]$ the number $[r] := p^r$ for a fixed prime number p . We recollect the standard definition of the group of p -adic integers which we will denote in this thesis by \mathcal{Z}_p .

Definition 1.45 (p -Adic Integers). Let p be a fixed prime number. Consider the non-negative integers \mathbb{N} with the natural ordering “ \leq ” as a directed set. For $i \leq j$ let

$$h_i^j: \mathbb{Z}_{[j]} \rightarrow \mathbb{Z}_{[i]}$$

be the canonical projection. Then $\{\mathbb{Z}_{[i]}, h\}$ forms an inverse system directed by \mathbb{N} . Its limit is called the p -adic integers, in symbols

$$\mathcal{Z}_p := \varprojlim \mathbb{Z}_{[i]}.$$

Finite groups with the discrete topology are compact groups. We use this fact to topologize the group \mathcal{Z}_p which then becomes a compact group by Proposition 1.20.

We denote by h_i the projections $h_i: \mathcal{Z}_p \rightarrow \mathbb{Z}_{[i]}$ which are continuous and we denote by $\mathbf{Z}_i := \ker(h_i)$ the kernels of those homomorphisms. Since the group $\mathbb{Z}_{[i]}$ is a discrete group, the subset $\{0\}$ is an open subset of $\mathbb{Z}_{[i]}$ and thus $\mathbf{Z}_i = h_i^{-1}(\{0\})$ is an open subgroup of the group of \mathcal{Z}_p .

If $i \leq j$ and $x \in \mathbf{Z}_j$, then $h_i(x) = h_i^j(h_j(x)) = h_i^j(0) = 0$ and thus $x \in \mathbf{Z}_i$, that is $\mathbf{Z}_j \subset \mathbf{Z}_i$. Therefore we get a descending series of open subgroups

$$\mathcal{Z}_p = \mathbf{Z}_0 \supset \mathbf{Z}_1 \supset \mathbf{Z}_2 \supset \mathbf{Z}_3 \supset \dots$$

with $\mathbb{Z}_{[i]} \cong \mathcal{Z}_p / \ker h_i = \mathcal{Z}_p / \mathbf{Z}_i$.

The group of p -adic integers belongs to the family of compact, totally disconnected abelian groups. More specifically one gets from [RZ00, p. 11, Theorem 1.1.12] the following result.

Proposition 1.46. *Let G be a topological group. Then G is a compact and totally disconnected group if and only if G is isomorphic as topological groups to an inverse limit of some finite groups G_i . Furthermore the groups G_i can be chosen to be all abelian if and only if G is abelian. \square*

One property of locally compact and totally disconnected topological groups is that the collection of open subgroups form a neighborhood system of the identity element, see for example [Pon66, p. 131].

In the compact case one has even the following more specific result. By the previous proposition we know that if G is a compact and totally disconnected group, then G can be seen as the limit

$$G = \varprojlim G_i$$

of some inverse system $\{G_i, \varphi\}$ of finite groups directed by some indexset I . Denote by $\tilde{G} := \prod G_i$ the direct product of the groups and denote by $\varphi_i: \tilde{G} \rightarrow G_i$ the projections. Let $U \subset G$ be an open neighborhood of the identity element e . Then $U = \tilde{U} \cap G$ for some open set $\tilde{U} \subset \tilde{G}$. By the definition of the product topology we have

$$\tilde{U} = \prod U_i$$

where the U_i are open subsets of G_i such that $U_i = G_i$ for all $i \in I$ but finitely many exceptions. Denote by $J \subset I$ the finite set of indices such that $U_i \neq G_i$. Then

there exists a $i_0 \in I$ such that $j \leq i_0$ for every $j \in J$. Then we have clearly that $\ker(\varphi_{i_0}) \subset U$. Since $\ker(\varphi_i)$ are open subsets of G it follows that those sets form a neighborhood system of the identity element. Thus we have shown the following result.

Proposition 1.47. *Let $G = \varprojlim G_i$ be a limit of an inverse system of finite groups directed by some index set I and with projections $\varphi_i: G \rightarrow G_i$. Then the set $\{\ker(\varphi_i)\}_{i \in I}$ forms a neighborhood system of the identity element. \square*

In particular we have that the collection $\{\mathbf{Z}_i\}_{i \in \mathbb{N}}$ forms a neighborhood system of the identity element of the p -adic integers \mathcal{Z}_p .

Let $H \leq \mathcal{Z}_p$ be a non-trivial closed subgroup. Then there exists a minimal $i \in \mathbb{N}$ such that there exists a $x \in H$ with $h_{i+1}(x) \neq 0$. Then $H \subset \mathbf{Z}_i$. On the other hand it follows that $\mathbf{Z}_i = \overline{\langle x \rangle} \subset H$. Thus we have the equality $\mathbf{Z}_i = H$. Therefore we know all closed subgroups of \mathcal{Z}_p , that is we have the following classification of the closed subgroups of \mathcal{Z}_p .

Proposition 1.48. *Let $H \leq \mathcal{Z}_p$ be a closed subgroup. Then either $H = \mathbf{Z}_i$ for some $i \in \mathbb{N}$ or $H = \{0\}$ is the trivial group. \square*

We know already that the subgroups \mathbf{Z}_i are open subgroups. On the other hand it follows that there are no other open subgroups in \mathcal{Z}_p . Thus we have the following classification of the open subgroups of \mathcal{Z}_p .

Proposition 1.49. *Let $H \leq \mathcal{Z}_p$ be an open subgroup. Then $H = \mathbf{Z}_i$ for some $i \in \mathbb{N}$. \square*

The open subgroups of \mathcal{Z}_p enjoy a simple structure, namely we have the following result.

Proposition 1.50. *The groups \mathbf{Z}_i are topologically isomorphic. In particular*

$$\mathcal{Z}_p \cong \mathbf{Z}_i$$

for every $i \in \mathbb{N}$ and this isomorphism is given by $x \mapsto p^i \cdot x$. \square

8. Homology Dimension

Homology dimension is in this thesis the natural choice for a dimension as its definition uses the Čech homology groups. We will give in this section its definition and we will collect a few useful results which we will need in this thesis. In the end we will state how homology dimension relates to the Lebesgue covering dimension.

Definition 1.51 (Homology Dimension for Compact Spaces). Let X be a compact space and let \mathcal{G} be a compact abelian group. Let $n \geq -1$ be an integer. Then the space X is said to have *homology dimension* $\leq n$ (with respect to the coefficient group \mathcal{G}), if $\check{H}_k(M, N; \mathcal{G}) = 0$ for every $k > n$ and every compact pair (M, N) with $M \subset X$. If X is of homology dimension $\leq n$ but not of homology dimension $\leq n - 1$, then X is said to have homology dimension n . If X is not of homology dimension $\leq n$ for any integer, then X is said to have infinite homology dimension. In symbols we denote the fact that X has the homology dimension n or infinite (with respect to \mathcal{G}) by $\text{hd}_{\mathcal{G}} X := n$ or $\text{hd}_{\mathcal{G}} X := \infty$ respectively.

Note that one can show – see [Ale47] – that $\text{hd}_{\mathcal{G}} X \leq \text{hd}_{\mathcal{R}} X$ for any coefficient group \mathcal{G} . This and the fact that in this thesis we will use mainly \mathcal{R} as the coefficient group for the homology groups makes it the natural choice to omit \mathcal{R} in the notation of homology dimension. That is, we will write $\text{hd} X$ instead of $\text{hd}_{\mathcal{R}} X$.

Yang extends the homology dimension to locally compact spaces in the following way:

Definition 1.52 (Homology Dimension for Locally Compact Spaces). Let X be a locally compact space and \mathcal{G} a compact group. Then its *homology dimension* with respect to the coefficient group \mathcal{G} is

$$\text{hd}_{\mathcal{G}} X := \sup\{\text{hd}_{\mathcal{G}} K : K \subset X \text{ compact}\}.$$

Referring to [Coh54] Yang states the following two results.

Proposition 1.53. *Let X be a locally compact space with finite homology dimension. Then there exists a point $x \in X$ such that for all neighborhoods U of x we have the equality $\text{hd} U = \text{hd} X$.* \square

Proposition 1.54. *Let X be a locally compact space and $A \subset X$ closed. Then $\text{hd}_X = \max(\text{hd} A, \text{hd}(X \setminus A))$.* \square

Recall that the locally closed subsets of a topological space are precisely those which can be written as the intersection of an open and a closed subset of X (see [Bou89, p. 38, Proposition 5]). A simple calculation verifies the following corollary to the above proposition.

Corollary 1.55. *Let $B \subset X$ be a locally closed subset of a locally compact space X . Then $\text{hd} X = \max(\text{hd} B, \text{hd}(X \setminus B))$.* \square

In particular $\text{hd} B \leq \text{hd} X$ for every locally compact $B \subset X$, and if $\text{hd} B < \text{hd} X$, then necessarily $\text{hd} X = \text{hd}(X \setminus B)$.

Let $\alpha: V_\alpha \rightarrow X$ be a covering of the space X and $n \geq -1$ an integer. Then α is said to be of *order* $\leq n$, if the set $\{v \in V_\alpha : x \in \alpha_v\}$ has at most $n + 1$ elements.

Definition 1.56 (Lebesgue Covering Dimension for Compact Spaces). Let X be a compact space and let $n \geq -1$. We say that X has *covering dimension* $\leq n$ if every open covering of X can be refined by a covering of order $\leq n$. If X is of dimension $\leq n$ but not of dimension $\leq n - 1$, then X is said to have covering dimension n . If X does not have covering dimension $\leq n$ for any integer $n \geq -1$, then X is said to have infinite covering dimension. In symbols we denote the fact that X has the covering dimension n or infinite by $\dim X = n$ or $\dim X = \infty$ respectively.

The extension of the Lebesgue covering dimension to locally compact spaces is then done in a similar fashion as the concept of homology dimension is extended from the compact to the locally compact case.

Definition 1.57 (Lebesgue Covering Dimension for Locally Compact Spaces). Let X be a locally compact space. Then its *covering dimension* is

$$\dim X := \sup\{\dim K : K \subset X \text{ compact}\}.$$

Now Yang refers to [Ale47] for the next results which gives a relation between the homology dimension and the covering dimension of a locally compact space.

Proposition 1.58. *Let X be a locally compact space of finite covering dimension. Then $\text{hd} X = \dim X$.* \square

In particular, if X is a locally compact space with $\text{hd} X = n$, then either $\dim X = n$ or $\dim X = \infty$.

If X is a locally compact space where a group G acts as a topological transformation group, then a natural question is in which way the homology dimension of X and the homology dimension of its orbit space X/G relate to each other. In the case that G is a finite group there exists a simple relation. Yang states in [Yan60, p. 204] the following result.

Proposition 1.59. *Let X be a locally compact G -space with finite homology dimension. If G is a finite group then $\text{hd } X/G = \text{hd } X$. \square*

But as already outlined in the introduction this result does not hold true in the general case where G is an arbitrary transformation group. For example Yang proves in [Yan60] that for an effective action of the p -adic integers \mathcal{Z}_p on a homology n -manifold X we have the equality

$$\text{hd}(X/\mathcal{Z}_p) = \text{hd}(X) + 2.$$

The following chapter is devoted to Yang's construction of a modified Smith theory and a few of its technical results which is the essential tool Yang uses in his article to prove this result.

CHAPTER 2

Special Homology Groups for p -Adic Actions

We will construct in this chapter the special homology groups for p -adic actions on a compact space X as Yang does in his paper [Yan60]. The construction will be done in steps, beginning from special homology groups for simplicial G -complexes where G is a cyclic p -group. Then we will extend this definition to Čech homology theory where still the action is given by a cyclic p -group. And then we will see how to use the fact that the group \mathbb{Z}_p of p -adic integers is the inverse limit of cyclic p -groups in order to extend the definition to Čech homology theory where the action is given by the p -adic integers.

In the original P.A. Smith theory the special homology groups had been constructed using a cyclic group C_p of p elements (p prime) as a coefficient group. For our purpose we need to deal with cyclic coefficient groups of order p^r for different values $r \in \mathbb{N}$ at the same time. Yang solves this problem by using the reals modulo the integers $\mathcal{R} := \mathbb{R}/\mathbb{Z}$ as the coefficient group for the homology groups. Thus in order to simplify the notation we will assume that the coefficient groups in the chain complexes and homology groups used in this chapter are \mathcal{R} in case they are omitted in the notation. That is, we will write for example $C_k(K, L)$ but actually mean $C_k(K, L; \mathcal{R})$ and $H_k(X, A)$ will stand for $H_k(X, A; \mathcal{R})$. Note that this agreement differs from the agreement in Section 1 and Section 5 of the previous chapter.

In this chapter p will be a fixed prime number. Recall that in order to simplify the notation we use the symbol $[r]$ to denote the r -th power of p , that is $[r] := p^r$.

1. Definition for Simplicial G -Complexes

In this section we will consider the action of a cyclic p -group G on a simplicial complex, that is, we consider the cyclic group $G = C_{[r]}$ of p^r elements for some fixed $r \geq 0$. We shall use the multiplicative notation for G . Further we shall fix a generator g_0 of the group G .

In the group ring of G , denoted by $\mathbb{Z}[G]$, we define two special elements, namely

$$\sigma := 1 + g_0 + g_0^2 + \dots + g_0^{[r]-1} \quad \text{and} \quad \tau := 1 - g_0$$

Note that σ is by construction a G -invariant element and thus it follows immediately from the definition of τ that σ and τ are zero-divisors in $\mathbb{Z}[G]$, that is $\sigma\tau = 0$.

Let K be a regular simplicial G -complex. We consider the chain complex $C(K)$ of ordered simplices of K and coefficients in \mathcal{R} . The action of G on K makes the groups $C_k(K)$ into $\mathbb{Z}[G]$ modules and the above defined elements define two chain maps:

$$\begin{aligned} \sigma: C(K) &\rightarrow C(K), c \mapsto \sigma c \\ \tau: C(K) &\rightarrow C(K), c \mapsto \tau c \end{aligned}$$

Since σ and τ are zero-divisors in $\mathbb{Z}[G]$ it follows that we have the inclusions $\iota: \text{im } \tau \hookrightarrow \text{ker } \sigma$ and $\iota': \text{im } \sigma \hookrightarrow \text{ker } \tau$. Further we shall denote by ω and ω' the inclusions of $\text{ker } \sigma$ and $\text{ker } \tau$, respectively, into $C(K)$. With this notation the following result is then evident.

Lemma 2.1. *We have the following four short exact sequences of chain complexes:*

$$0 \longrightarrow \ker \sigma \xrightarrow{\omega} C(K) \xrightarrow{\sigma} \operatorname{im} \sigma \longrightarrow 0 \quad (2.2)$$

$$0 \longrightarrow \ker \tau \xrightarrow{\omega'} C(K) \xrightarrow{\tau} \operatorname{im} \tau \longrightarrow 0 \quad (2.3)$$

$$0 \longrightarrow \operatorname{im} \tau \xrightarrow{\imath} \ker \sigma \longrightarrow \ker \sigma / \operatorname{im} \tau \longrightarrow 0 \quad (2.4)$$

$$0 \longrightarrow \operatorname{im} \sigma \xrightarrow{\imath'} \ker \tau \longrightarrow \ker \tau / \operatorname{im} \sigma \longrightarrow 0 \quad (2.5)$$

Here the unnamed arrows in the last two sequences are the canonical projections. \square

We will soon find out that that $\operatorname{im} \sigma = \ker \tau$ and therefore the short exact sequence (2.5) will reduce to $0 \rightarrow \operatorname{im} \sigma \rightarrow \ker \tau \rightarrow 0$ where the middle arrow given by \imath' is the identity. But before we turn our attention to this issue we shall introduce a few more chain maps and results about them.

The simplicial projection $\pi: K \rightarrow K/G$ onto the orbit space defines a surjective chain map $\pi: C(K) \rightarrow C(K/G)$. For this chain map we have the following result.

Lemma 2.6. *The chain map $\sigma: C(K) \rightarrow \operatorname{im} \sigma$ factors through the chain map $\pi: C(K) \rightarrow C(K/G)$. That is, there exists a unique chain map $\kappa: C(K/G) \rightarrow \operatorname{im} \sigma$ such that $\sigma = \kappa\pi$.*

PROOF. It is enough to show this result for a chain $c \in C_k(K)$ of the form

$$c = \sum_{s \in S} \alpha_s s \quad (\text{where } \alpha_s \in \mathcal{R})$$

where $S = \{s, g_0 s, g_0^2 s, \dots, g_0^{[r]-1} s\}$ is the orbit of some ordered k -simplex of K . We shall call such a chain in the following an *elementary G -chain*. Then

$$\pi(c) = \alpha_S \pi(s) \quad \text{and} \quad \sigma(c) = \frac{|G|}{|S|} \alpha_S \sum_{s \in S} s$$

where $\alpha_S = \sum_{s \in S} \alpha_s$. Now the values $\pi(c)$ and $\sigma(c)$ depend only on the orbit S and α_S . Thus the assignment $\kappa(\pi(c)) := \sigma(c)$ is well defined. The surjectiveness of the chain map π ensures then that we get a chain map $\kappa: C_k(K/G) \rightarrow \operatorname{im} \sigma$ and by construction we get that the diagram

$$\begin{array}{ccc} C_k(K) & & \\ \downarrow \pi & \searrow \sigma & \\ C_k(K/G) & \xrightarrow{\kappa} & \operatorname{im} \sigma \end{array} \quad (2.7)$$

commutes.

It remains to show that κ is a chain map. We know already that π and σ are chain maps. By the surjectivity of π we have that for every k -chain $\bar{c} \in C_k(K/G)$ there exists a k -chain $c \in C_k(K)$ such that $\pi(c) = \bar{c}$. Then

$$\partial(\kappa(\bar{c})) = \partial(\kappa(\pi(c))) = \partial(\sigma(c)) = \sigma(\partial(c)) = \kappa(\pi(\partial(c))) = \kappa(\partial(\pi(c))) = \kappa(\partial(\bar{c}))$$

and thus $\kappa: C(K/G) \rightarrow \operatorname{im} \sigma$ is a chain map which has by construction the required properties. \square

Note that it follows from the surjectiveness of σ and π that the chain map κ is surjective, too.

Lemma 2.8. *We have the equality $\text{im } \tau = \ker \pi$.*

PROOF. “ $\text{im } \tau \subset \ker \pi$ ”: For any chain $c \in C(K)$ we have $\pi(c) = \pi(g_0 c)$. Thus $\pi \tau(c) = \pi(c - g_0 c) = \pi(c) - \pi(g_0 c) = 0$ and therefore $\text{im } \tau \subset \ker \pi$.

“ $\ker \pi \subset \text{im } \tau$ ”: On the contrary assume $c \in \ker \pi$. We need only to consider the case where c is an elementary G -chain. Assume that we can write

$$c = \alpha_0 s_0 + \dots + \alpha_k s_k \quad (\alpha_i \in \mathcal{R})$$

and that $g_0 s_i = s_{i+1}$ (for $0 \leq i < k$) and $g_0 s_k = s_0$. Since $\pi(c) = \alpha_0 + \dots + \alpha_k = 0$ it follows that we have necessarily

$$\alpha_k = -(\alpha_0 + \dots + \alpha_{k-1})$$

Define $c' := \alpha_0 s_0 + \dots + \alpha_{k-1} s_{k-1}$. Then we have by construction

$$\tau(c') = \alpha_0 s_0 + \dots + \alpha_{k-1} s_{k-1} - (\alpha_0 + \dots + \alpha_{k-1}) s_k = c$$

Therefore $c \in \text{im } \tau$ and we have shown that $\ker \pi \subset \text{im } \tau$. \square

We denote by $D(K)$ the image of $\ker \sigma$ in $C(K/G)$ under the projection π , that is $D(K) := \pi(\ker \sigma)$, and we denote by $\theta: D(K) \hookrightarrow C(K/G)$ the inclusion.

Corollary 2.9. *We have the following short exact sequences:*

$$0 \longrightarrow \text{im } \tau \xrightarrow{\omega} C(K) \xrightarrow{\pi} C(K/G) \longrightarrow 0 \quad (2.10)$$

$$0 \longrightarrow \text{im } \tau \xrightarrow{\iota} \ker \sigma \xrightarrow{\pi} D(K) \longrightarrow 0 \quad (2.11)$$

\square

Lemma 2.12. *The sequence*

$$0 \longrightarrow D(K) \xrightarrow{\theta} C(K/G) \xrightarrow{\kappa} \text{im } \sigma \longrightarrow 0 \quad (2.13)$$

is exact.

PROOF. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \sigma & \xrightarrow{\omega} & C(K) & \xrightarrow{\sigma} & \text{im } \sigma & \longrightarrow & 0 \\ & & \pi \downarrow & & \pi \downarrow & & \text{id} \downarrow & & \\ 0 & \longrightarrow & D(K) & \xrightarrow{\theta} & C(K/G) & \xrightarrow{\kappa} & \text{im } \sigma & \longrightarrow & 0 \end{array} \quad (2.14)$$

with all the vertical arrows being epimorphisms. Then the exactness of the lower row follows from the exactness of the upper row by simple diagram chasing. \square

From the above diagram (2.14) we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \sigma / \text{im } \tau & \xrightarrow{\bar{\omega}} & C(K) / \text{im } \tau & \xrightarrow{\bar{\sigma}} & \text{im } \sigma & \longrightarrow & 0 \\ & & \bar{\pi} \downarrow \cong & & \bar{\pi} \downarrow \cong & & \text{id} \downarrow & & \\ 0 & \longrightarrow & D(K) & \xrightarrow{\theta} & C(K/G) & \xrightarrow{\kappa} & \text{im } \sigma & \longrightarrow & 0 \end{array} \quad (2.15)$$

where the homomorphisms $\bar{\omega}$, $\bar{\sigma}$ and $\bar{\pi}$ are induced by ω , σ and π respectively. Then $\bar{\pi}$ is an isomorphism since $\text{im } \tau = \ker \pi$. Observe that the upper row of this diagram represents the sequence of isomorphisms

$$\frac{C(K) / \text{im } \tau}{\ker \sigma / \text{im } \tau} \cong C(K) / \ker \sigma \cong \text{im } \sigma.$$

Note that the above results, especially diagram (2.15) with its isomorphisms, is due to the exact sequences of Lemma 2.8 and the fact that the chain map σ factors through $\pi: C(K/G) \rightarrow \text{im } \sigma$ by Lemma 2.6.

For $\text{im } \sigma$ we shall now show a similar equality as we have shown for $\text{im } \tau$ in Lemma 2.8. We begin with the observation which follows directly from the definition of σ .

Lemma 2.16. *Let $c \in C(K)$ be a G -invariant element. Then $\sigma(c) = [r]c$ \square*

If c is a G -invariant k -chain of $C(K)$ then so is $c' := 1/[r]c$. Thus we have by the above result the equality $c = \sigma(c')$ which shows that $c \in \text{im } \sigma$. On the other hand, since g_0 is a generator of G , we have from the definition of the chain map τ that $c \in \ker \tau$ if and only if c is G -invariant. Therefore we have that $\ker \tau \subset \text{im } \sigma$. Since we know already that $\text{im } \sigma \subset \ker \tau$ this proves the next result.

Lemma 2.17. *We have the equality $\text{im } \sigma = \ker \tau$. In particular the inclusion $\iota': \text{im } \sigma \hookrightarrow \ker \tau$ is the identity. \square*

Next we turn our attention to the chain complex $D(K)$ which we want to describe more precisely. Therefore we need to have a look at the fixed point sets of the subgroups of G .

Denote for $0 \leq i \leq r$ by H_i the subgroups of G generated by $g_0^{[i]}$. Then those groups form a descending series

$$G = H_0 \supset H_1 \supset \dots \supset H_{r-1} \supset H_r = 0 \quad (2.18)$$

of subgroups of G , and this describes the subgroup structure of G exhaustively. We denote by $L_i \subset K$ the fixed point sets of those subgroups, which form then an ascending series

$$L_0 \subset L_1 \subset \dots \subset L_{r-1} \subset L_r = K$$

of simplicial complexes. Since G is an abelian group the fixed point sets L_i are all G -invariant and since the action of G on K is assumed to be regular the action of G on the L_i is regular, too. Thus the orbit spaces L_i/G have the structure of simplicial complexes. These simplicial complexes form an ascending series

$$L_0/G \subset L_1/G \subset \dots \subset L_{r-1}/G \subset L_r/G = K/G$$

of simplicial subcomplexes of K/G , too.

In the following we denote for every non-negative integer i by Z_i the cyclic subgroup of \mathcal{R} with i elements.

Let $s \in K_\circ$ be an ordered simplex of K with H_i being its isotropy group. Then the orbit S of s under the action of G has precisely $|G|/|H_i| = [r]/[r-i] = [i]$ elements. Thus if c is an elementary G -chain of $C(K)$ of the form $c = \sum_{s \in S} \alpha_s s$ then

$$\sigma(c) = [r-i] \alpha_S \sum_{s \in S} s = 0$$

if and only if $\alpha_S \in Z_{[r-i]}$. Thus $c \in \ker \sigma$ if and only if $\pi(c) \in C(L_i/G; Z_{[r-i]})$. From this observation together with the fact, that every chain of $C(K)$ can be written as a finite sum of elementary G -chains, follows then the next result about the chain complex $D(K)$.

Lemma 2.19.

$$D(K) = \sum_{k=0}^{r-1} C(L_k/G; Z_{[r-k]}) \quad \square$$

Note that the sum of the right hand side of this equation is not the direct sum!

Next we shall pass to the homology groups of the above chain complexes. The homology groups of the images and kernels of the chain maps σ and τ are of special interest and this leads to the following definition.

Definition 2.20 (Special Homology Groups). *The k -th special homology groups of the simplicial G -complex K are the following four homology groups:*

$$\begin{aligned}\bar{H}_k^\sigma(K) &:= H_k(\text{im } \sigma) & H_k^\sigma(K) &:= H_k(\ker \sigma) \\ \bar{H}_k^\tau(K) &:= H_k(\text{im } \tau) & H_k^\tau(K) &:= H_k(\ker \tau)\end{aligned}$$

Moreover we shall denote in this thesis by $I_k(K)$ the homology groups of the chain complex $D(K)$.

Remark 2.21. Note that the definition of $\bar{H}_k^\sigma(K)$ and $\bar{H}_k^\tau(K)$ in this thesis differs from the way it is defined in [Yan60], where Yang defines $\bar{H}_k^\sigma(K) := H_k(\text{im } \tau)$ and $\bar{H}_k^\tau(K) := H_k(\text{im } \sigma)$!

Proposition 2.22. *There exist the following long exact sequences:*

$$\dots \longrightarrow H_k^\sigma(K) \xrightarrow{\omega_*} H_k(K) \xrightarrow{\sigma_*} \bar{H}_k^\sigma(K) \longrightarrow H_{k-1}^\sigma(K) \xrightarrow{\omega_*} \dots \quad (2.23)$$

$$\dots \longrightarrow H_k^\tau(K) \xrightarrow{\omega'_*} H_k(K) \xrightarrow{\tau_*} \bar{H}_k^\tau(K) \longrightarrow H_{k-1}^\tau(K) \xrightarrow{\omega'_*} \dots \quad (2.24)$$

$$\dots \longrightarrow \bar{H}_k^\tau(K) \xrightarrow{\omega_* \iota_*} H_k(K) \xrightarrow{\pi_*} H_k(K/G) \longrightarrow \bar{H}_{k-1}^\tau(K) \xrightarrow{\omega_* \iota_*} \dots \quad (2.25)$$

$$\dots \longrightarrow \bar{H}_k^\tau(K) \xrightarrow{\iota_*} H_k^\sigma(K) \xrightarrow{\pi_*} I_k(K) \longrightarrow \bar{H}_{k-1}^\tau(K) \xrightarrow{\iota_*} \dots \quad (2.26)$$

and

$$\dots \longrightarrow I_k(K) \xrightarrow{\theta_*} H_k(K/G) \xrightarrow{\kappa_*} \bar{H}_k^\sigma(K) \longrightarrow I_{k-1}(K) \xrightarrow{\theta_*} \dots \quad (2.27)$$

where the non-labeled homomorphisms are the appropriate connecting homomorphisms.

Furthermore the inclusion $\iota': \text{im } \sigma \rightarrow \ker \tau$ induces the identity

$$\iota'_*: \bar{H}_k^\sigma(K) \xrightarrow{=} H_k^\tau(K)$$

PROOF. The first two long exact sequences are obtained from the short exact sequences (2.2) and (2.3). The next two sequences are the long exact homology sequences induced by the short exact sequences of Corollary 2.9. The last long exact sequence is the long homology sequence of the short exact sequence of Lemma 2.12. And finally since $\iota': \text{im } \sigma \rightarrow \ker \tau$ is by Lemma 2.17 the identity the induced homomorphism ι'_* is the identity, too. \square

Lemma 2.28. *We have the following commutative diagram*

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_k^\sigma(K) & \xrightarrow{\omega_*} & H_k(K) & \xrightarrow{\sigma_*} & \bar{H}_k^\sigma(K) & \longrightarrow & \dots \\ & & \pi_* \downarrow & & \pi_* \downarrow & & \text{id} \downarrow & & \\ \dots & \longrightarrow & I_k(K) & \xrightarrow{\theta_*} & H_k(K/G) & \xrightarrow{\kappa_*} & \bar{H}_k^\sigma(K) & \longrightarrow & \dots \end{array} \quad (2.29)$$

PROOF. This is a consequence of the commutativity of the diagram (2.14). \square

From the definition of σ we get the analogous result to Lemma 2.16 for the corresponding homology groups. We have the following sequence of homomorphisms:

$$H_k(K) \xrightarrow{\sigma_*} \bar{H}_k^\sigma(K) \xrightarrow{\iota'_*} H_k^\tau(K) \xrightarrow{\omega'_*} H_k(K)$$

The composite of these homomorphisms must agree with the homomorphism

$$\sigma_*: H_k(K) \rightarrow H_k(K)$$

which is induced on the chain level by $\sigma: C(K) \mapsto C(K)$. Now if e is a G -invariant element of $H_k(K)$, then

$$\sigma_*(e) = e + g_{0*}(e) + g_{0*}^2(e) + \dots + g_{0*}^{[r]-1}(e) = [r]e$$

and therefore we have the shown

Lemma 2.30. *Let $e \in H(K)$ be a G -invariant element. Then $\omega'_* \iota'_* \sigma_*(e) = [r]e$. \square*

Next we consider the following composite of chain maps.

$$C(K/G) \xrightarrow{\kappa} \text{im } \sigma \xrightarrow{\iota'} \ker \tau \xrightarrow{\omega'} C(K) \xrightarrow{\pi} C(K/G)$$

We want to prove that a result similar to Lemma 2.16 holds for elements of the chain complex $C(K/G)$ and sequence of maps above. Note that every chain of $C(K/G)$ is G -invariant and therefore we may omit to explicitly mention this requirement. Thus the result we want to prove is the following.

Lemma 2.31. *Let $\bar{c} \in C(K/G)$. Then $\pi \omega' \iota' \kappa(\bar{c}) = [r]\bar{c}$.*

PROOF. It is enough to verify the claim for a chain of the type $\bar{c} = \alpha \bar{s}$ where \bar{s} is a generator of $C(K/G)$, that is \bar{s} is an ordered simplex of K/G . Let s be an ordered simplex of K such that $\pi(s) = \bar{s}$ and S the orbit of s under the action of G . Then

$$c := \frac{\alpha}{|S|} \sum_{s' \in S} s'$$

is a G -invariant element of $C(K)$ such that by construction $\pi(c) = \bar{c}$. From Lemma 2.16 we get the equality $\sigma(c) = [r]c$. Then

$$\pi \omega' \iota' \kappa(\bar{c}) = \pi \omega' \iota' \kappa \pi(c) = \pi \omega' \iota' \sigma(c) = [r]\pi(c) = [r]\bar{c}. \quad \square$$

Corollary 2.32. *Let $e \in H(K/G)$. Then $\pi_* \omega'_* \iota'_* \kappa_*(e) = [r]e$. \square*

Note, that when we define the chain map $\eta: C(K/G) \rightarrow C(K)$ to be the composite $\eta := \omega' \iota' \kappa$ then we get the homomorphism which is known in the literature as the *transfer homomorphism*. We have then

$$\pi_* \eta_*: H(K/G) \rightarrow H(K/G), e \mapsto [r]e \quad (2.33)$$

which is the result of Corollary 2.32 and

$$\eta_* \pi_*: H(K) \rightarrow H(K), e \mapsto \sum_{g \in G} g(e) \quad (2.34)$$

which in the case that the element e is invariant under the action of G reduces to the result of Lemma 2.30.

2. Special Coverings

In the case of the Čech homology groups the link between the algebraic construction of the homology groups and the topological properties of the space X are open coverings and their nerves. In order to extend the definition of special homology groups to the Čech homology groups of a compact space X we need the existence of a cofinal subset of $\text{Cov}(X)$ such that the nerve of each covering in this subset is a regular G -complex. In order to topologize the homology groups we need the coverings to be finite. These requirements lead to the Definition 2.35 of *special coverings*. In the next section we will then use the results of this section to extend the definition of special homology groups to the Čech homology groups.

The exposition of the content in this section has been improved compared to Yang's article. There the result of Proposition 2.36 is only mentioned in a short note

without explanation of its importance [Yan60, p. 206], and furthermore essential details in the proof of Proposition 2.39 – in particular this is the explicit verification of the properties of a special covering – are left out [Yan60, p. 203].

Throughout this section X will be a compact G -space where G is a finite cyclic p -group as introduced in the previous section. Recall the descending sequence of subgroups of G ,

$$G = H_0 \supset H_1 \supset \dots \supset H_{r-1} \supset H_r = 0$$

as it has been defined in in the previous section.

The fixed point sets of those subgroups will be denoted in the following by F_i , that is $F_i := X^{H_i}$. Thus we get an ascending sequence

$$X^G = F_0 \subset F_1 \subset \dots \subset F_{r-1} \subset F_r = X$$

of closed subsets of X .

Yang introduces in his article [Yan60, p. 202] the following definition for a special covering.

Definition 2.35 (Special Covering). Let A be a closed G -invariant subspace of X . Let λ be a self-indexed covering of A in X . Then λ is said to be a *special covering*, if it satisfies the following four conditions:

- (1) The covering λ is finite and G -invariant.
- (2) For every $U \in \lambda$ and $g \in G$ follows from $gU \neq U$ that $\overline{gU} \cap \overline{U} = \emptyset$.
- (3) Let $U \in \lambda$ and $g \in G$. If there exist a $V \in \lambda$ which is not g -invariant, then from $U \cap V \neq \emptyset$ and $U \cap gV \neq \emptyset$ it follows that necessarily $gU = U$.
- (4) For any H_i -invariant members U_0, \dots, U_k of λ with $U_0 \cap \dots \cap U_k \neq \emptyset$, the intersection $U_0 \cap \dots \cap U_k \cap A \cap F_i \neq \emptyset$, too.

The set of all special open coverings of A in X is denoted by $\text{Cov}_{\text{sp}}(A; X)$. The set of all special open coverings of X in itself is denoted by $\text{Cov}_{\text{sp}}(X)$.

Note that in the case of $G = \mathbb{Z}_p$ this definition is equivalent to the definition by Smith given in [Smi38].

Proposition 2.36. *Let λ be a special covering of A in X . Then the action of G on X defines a regular action of G on the nerve K_λ .*

Some preparative work is needed before we can begin to prove this proposition. We state the following two two auxiliary results.

Lemma 2.37. *A simplicial G -complex K satisfies property P_1 if and only if $v, gv \in s$ implies that $gv = v$ for all $s \in K$ and $g \in G$.*

PROOF. “ \Rightarrow ”: Assume that K satisfies property P_1 . Let v be a vertex of K and $g \in G$. Assume that $s = \{v, gv\}$ is a simplex of K . Then also $g^{-1}s = \{g^{-1}v, v\}$ is a simplex of K and therefore $v \in s \cap g^{-1}s$. Then by the property P_1 it follows that $v = gv$.

“ \Leftarrow ”: Let $v \in s \cap g^{-1}s$. Then v and gv belong both to s and thus by assumption $gv = v$. Thus K satisfies property P_1 . \square

Lemma 2.38. *Assume that K is a G -complex satisfying in addition to property P_1 also the following property: for any vertex u of K and any $g \in G$, if there exists a vertex v such that $\{u, v\}, \{u, gv\} \in K$ but $\{v, gv\} \notin K$, then $gu = u$. Then K is a regular G -complex.*

PROOF. Observe that if the conditions are satisfied for G then they are also satisfied for every subgroup $H \subset G$. Thus it is enough to show that the above condition implies that the G -complex satisfies property P_2 , because then K satisfies also property P_2 as a H -complex for every subgroup $H \subset G$.

Now the claim follows from repeated use of the following result: let $s = \{u_0, \dots, u_k\}$ and $t = \{v_0, \dots, v_k\}$ be two k -simplices of K and assume that there exist $g_i \in G$ such that $u_i = g_i v_i$ ($0 \leq i < k$). If for some $r < k$ there exists a $g \in G$ such that $u_i = g v_i$ for all $0 \leq i \leq r$, then g can be chosen such that $u_i = g v_i$ for all $0 \leq i \leq r+1$.

To verify this result we may assume that u_{r+1} is not $g g_{r+1}^{-1}$ -invariant, as otherwise $u_{r+1} = g g_{r+1}^{-1} u_{r+1} = g v_{r+1}$ and there is nothing to prove.

Let $0 \leq i \leq r$ and observe that since $s \in K$ also its face $\{u_i, u_{r+1}\}$ is a 1-simplex in K . Further, since $g t \in K$ it follows that also its face $\{g v_i, g v_{r+1}\} = \{u_i, g g_{r+1}^{-1} u_{r+1}\}$ is a 1-simplex in K . Finally since K is assumed to satisfy property P_1 and since u_{r+1} is not $g g_{r+1}^{-1}$ -invariant it follows that $\{u_{r+1}, g g_{r+1}^{-1} u_{r+1}\}$ is *not* a simplex of K (Lemma 2.37). Thus from the assumed additional property, applied to $\{u_i, u_{r+1}\} \in K$, $\{u_i, g g_{r+1}^{-1} u_{r+1}\} \in K$ but $\{u_{r+1}, g g_{r+1}^{-1} u_{r+1}\} \notin K$ it follows that u_i is $g g_{r+1}^{-1}$ -invariant and therefore also $g_{r+1} g^{-1}$ -invariant. This holds for any $0 \leq i \leq r$. Thus we have that $u_i = g v_i = g_{r+1} g^{-1} g v_i = g_{r+1} v_i$ for $0 \leq i \leq r$ and the claim follows since we have by construction $u_{r+1} = g_{r+1} v_{r+1}$. \square

PROOF OF PROPOSITION 2.36. It follows from the second condition for special coverings that whenever U and gU are two vertices of K_λ belonging to the same simplex, then necessarily $U = gU$. Thus it follows from Lemma 2.37 that K_λ satisfies property P_1 .

The third condition for special coverings provides exactly the missing requirements for Lemma 2.38 and applying this lemma concludes the proof. \square

Thus we have shown that the nerve of every special covering is a regular G -complex. Next we will show that every covering of a compact space can be refined by a special covering, that is

Proposition 2.39. *Let A be a closed G -invariant subset of X and α a covering of X . Then there exists an open special covering of A in X refining α .*

Before we prove this proposition we need to do some preparatory work.

Lemma 2.40. *Let X be a normal space and A a closed subset of X . Let α be an open covering of A in X and assume that $\lambda = \{W_0, \dots, W_m\}$ is a finite closed covering of A in itself refining α . Then there exists a finite closed covering $\mu = \{E_0, \dots, E_m\}$ of A in X refining α such that*

- (1) $W_i \subset \text{Int } E_i$ for every $0 \leq i \leq m$, and
- (2) $W_{i_0} \cap \dots \cap W_{i_k} \neq \emptyset$ if and only if $E_{i_0} \cap \dots \cap E_{i_k} \neq \emptyset$.

Further, if A and λ are both G -invariant then μ can be chosen to be G -invariant, too.

PROOF. The idea of the proof is to convert the covering λ step by step into the closed covering of A in X which has the desired properties. We begin by setting $\mu := \{W_0, \dots, W_m\}$.

Since X is a normal space there exists a closed subset E which is contained in some element of α such that $W_0 \subset \text{Int } E$.

Let $\{W_{i_1}, \dots, W_{i_k}\}$ be a subset of μ and set $B := W_{i_1} \cap \dots \cap W_{i_k}$. Assume that $B \neq \emptyset$, $W_0 \cap B = \emptyset$ but $E \cap B \neq \emptyset$. Since W_0 and B are disjoint closed subsets of the normal space X there exists a closed set E' such that $W_0 \subset \text{Int } E'$ and $E' \cap B = \emptyset$. We replace E by $E \cap E'$ and observe that for this new E we still have $W_0 \subset \text{Int } E$ but now also $E \cap B = \emptyset$. We repeat this for all finite subsets $\{W_{i_1}, \dots, W_{i_k}\} \subset \mu$ and since μ is finite this construction yields after finitely many steps a set E such that $W_0 \subset \text{Int } E$ and such that from $W_0 \cap W_{i_1} \cap \dots \cap W_{i_k} = \emptyset$

it follows that $E \cap W_{i_1} \cap \dots \cap W_{i_k} = \emptyset$, too. Finally we replace in μ the set W_0 by this set E .

Repeating this replacement for the remaining W_i 's ($1 \leq i \leq k$) in μ we reach after finitely many steps the desired covering of A in X .

If λ and A are both G -invariant then it follows from a standard argument that μ can be chosen to be G -invariant. \square

Lemma 2.41. *Let X be a normal space and A a closed subset of X . Let α be an open covering of A in X and assume that $\lambda = \{W_0, \dots, W_m\}$ is a finite closed covering of A in X refining α . Then there exists a finite open covering $\mu = \{U_0, \dots, U_m\}$ of A in X refining α such that*

- (1) $W_i \subset U_i$ for every $0 \leq i \leq m$, and
- (2) $W_{i_0} \cap \dots \cap W_{i_k} \neq \emptyset$ if and only if $U_{i_0} \cap \dots \cap U_{i_k} \neq \emptyset$.

Further, if A and λ are both G -invariant then μ can be chosen to be G -invariant, too.

PROOF. In the analogous way as in the proof of the previous Lemma 2.40 one constructs closed E_i 's such that for every $W_i \in \lambda$ we have $W_i \subset \text{Int } E_i$. Then $\mu := \{U_0, \dots, U_m\}$ with $U_i := \text{Int } E_i$ is the desired covering.

If λ and A are both G -invariant then it follows again by a standard argument that μ can be chosen to be G -invariant. \square

With the help of Lemma 2.40 and Lemma 2.41 we conclude

Lemma 2.42. *Let α be an open covering of A in X and assume that there exists a closed special covering λ of A in itself which refines α . Then there exists an open special covering μ of A in X which refines α and which is refined by λ . \square*

Thus the claim of Proposition 2.39 boils down to the question whether we can find for an arbitrary open covering α of A in X a closed special covering λ of A in itself which is refining α . Since any open cover α of A in X defines an open covering $\alpha' := \{U \cap A : U \in \alpha\}$ of A in itself it is enough to prove the following result in order to prove Proposition 2.39.

Proposition 2.43. *Let X be a compact G -space. Then for any open covering α of X in itself there exists a closed special covering λ of X in itself refining α .*

We need to add one more technical result which will be used in the proof of Proposition 2.43.

Lemma 2.44. *Let α be an open covering of X and assume that λ is a closed covering of F_{i-1} in X refining α . Then for any $x \in F_i \setminus F_{i-1}$ there exists an H_i -invariant neighborhood U_x of x contained in some element of α such that the sets $\{g\overline{U_x} : g \in G\}$ are pairwise disjoint and such that U_x intersects with an element $V \in \lambda$ if and only if $x \in V$.*

PROOF. Since λ is a finite closed covering the set

$$Y := \bigcup \{V \in \lambda : x \notin V\}$$

is closed in X . Then by construction for any neighborhood U of x which is contained in the complement of Y holds: U has a non-empty intersection with an element $V \in \lambda$ if and only if $x \in V$.

We denote by $\pi: X \rightarrow X/H_i$ the canonical projection onto the orbit space. Since $x \notin Y$ and x is a fixed point of the action of H_i we get that $\pi(x) \notin \pi(Y)$. Thus $U_0 := \pi^{-1}(X/H_i \setminus \pi(Y))$ is by construction an H_i -invariant open neighborhood of x which has an empty intersection with Y .

Set $\bar{x} := \pi(x)$. Let $U_\alpha \in \alpha$ be an element such that $x \in U_\alpha$. Since π is a closed map it follows that $\pi(X \setminus U_\alpha)$ is closed in X/H_i . Furthermore $\bar{x} \notin \pi(X \setminus U_\alpha)$ and

thus we may choose an open neighborhood \tilde{U}_1 of \bar{x} contained in $X/H_i \setminus \pi(X \setminus U_\alpha)$. Then $U_1 := \pi^{-1}(\tilde{U}_1)$ is an open H_i -invariant neighborhood of x in X and contained in U_α , an element of α .

Since X/H_i is a normal G/H_i -space it follows that we may choose for every $\bar{g} \in G/H_i$ a closed neighborhood $W_{\bar{g}}$ of $\bar{g}\bar{x}$ such that the $W_{\bar{g}}$ are pairwise disjoint. Then

$$W := \bigcap_{\bar{g} \in G/H_i} \bar{g}^{-1}W_{\bar{g}}$$

is a closed neighborhood of \bar{x} such that the sets $\bar{g}W$ for $\bar{g} \in G/H_i$ are all pairwise disjoint. Then by construction

$$U_x := \pi^{-1}(\text{Int } W) \cap U_0 \cap U_1$$

is a H_i -invariant open neighborhood of x satisfying all the requirements of the lemma. \square

PROOF OF PROPOSITION 2.43. We shall construct inductively a sequence of closed special coverings λ_i ($-1 \leq i \leq r$) of F_i in X ordered by inclusion

$$\lambda_{-1} \subset \lambda_0 \subset \lambda_1 \subset \dots \subset \lambda_{r-1} \subset \lambda_r$$

where each λ_i refines the covering α . Then λ_r is a closed special covering of X refining α .

For $i = -1$ the only choice we have is to set $\lambda_{-1} := \emptyset$. Thus we may assume that $i \geq 0$ and that λ_{i-1} is a closed special covering of F_{i-1} in X refining α . We denote by X_i the space

$$X_i := F_i \setminus \left(\bigcup_{U \in \lambda_{i-1}} \text{Int } U \right).$$

Then $X_i \subset F_i$ is a closed subset of X , possibly empty. If X_i is empty we set $\lambda_i := \lambda_{i-1}$ and λ_i is a special covering of F_i in X refining α and we are done with the construction step.

Thus we may assume that X_i is not empty. For every $x \in X_i$ we choose an H_i -invariant neighborhood U_x satisfying the requirements described in Lemma 2.44 with λ replaced by λ_{i-1} . Then $\beta := \{U_x : x \in X_i\}$ is an open covering of X_i in X refining α . It follows from Lemma 1.31 that there exists an open covering γ of X_i in X which star-refines β .

Fix a representative system $M \subset X_i$ for the orbit space X_i/G . For every $x \in M$ choose an H_i -invariant closed neighborhood D_x of x such that for every $g \in G$ the set gD_x is contained in some element of γ . If $x \in X_i \setminus M$, then choose a $g \in G$ such that $g^{-1}x \in M$ and define $D_x := g(D_{g^{-1}x})$. Since the D_x are H_i -invariant it follows that this definition does not depend on the choice of g and thus $\{D_x : x \in X_i\}$ forms a G -invariant closed covering of X_i in X which refines γ . By the compactness of X_i we may choose a *finite* G -invariant $S \subset X_i$ such that $\{D_x : x \in S\}$ is a closed covering of X_i in X . Using similar reasoning as in the proof of Lemma 2.40 we convince ourselves that we can choose the D_x 's in such a way that

$$D_{x_0} \cap \dots \cap D_{x_k} \neq \emptyset \quad \text{if and only if} \quad D_{x_0} \cap \dots \cap D_{x_k} \cap F_i \neq \emptyset \quad (*)$$

for every finite subset $\{x_0, \dots, x_k\} \subset S$.

Then for every $x \in S$ we define the closed set

$$E_x := D_x \cup \{y \in S : D_y \cap D_x \neq \emptyset\}$$

and with this notation we define further

$$\lambda_i := \lambda_{i-1} \cup \{E_x : x \in S\}.$$

which is then by construction a closed covering of F_i in X .

By assumption any element $U \in \lambda_{i-1}$ is contained in some element of α . If $U \in \lambda_i \setminus \lambda_{i-1}$, then $U = E_x$ for some $x \in S$. Now every $x' \in E_x$ is contained in some D_y such that $D_y \cap D_x \neq \emptyset$. Since γ is star-refining β and since $\{D_y : y \in S\}$ refines γ it follows that E_x is contained in some element of β and thus also in some element of α . Therefore the covering λ_i is refining the covering α .

We claim that λ_i is a special covering of F_i in X . We need to verify all the four requirements for special coverings:

- (1) By construction λ_i is finite. Furthermore it follows easily that $gE_x = E_{gx}$ for every $g \in G$ and thus from the G -invariance of S it follows that $\{E_x : x \in S\}$ is G -invariant, too. Therefore λ_i as a union of two G -invariant sets is itself again G -invariant. Hence we have shown that λ_i satisfies the first condition for special coverings.
- (2) Let $g \in G$ and assume that $U \in \lambda_i$ is not G -invariant. If $U \in \lambda_{i-1}$ then nothing needs to be proven. Thus we may assume that $U = E_x$ for some $x \in S$ and we have to show that $E_x \cap E_{gx} = \emptyset$. Now D_x is contained in some $V \in \beta$. Then $D_{gx} = gD_x \subset gV$ and since $V \cap gV = \emptyset$ we have that $D_x \cap D_{gx} = \emptyset$. Therefore

$$E_x \cap E_{gx} \subset \{y \in S : D_y \cap D_x \neq \emptyset\} \cap \{y \in S : D_y \cap D_{gx} \neq \emptyset\} \quad (**)$$

Assume towards a contradiction that the right hand side of this inclusion relation is not empty. Then there must exist a $y \in S$ such that $D_y \cap D_x \neq \emptyset$ and $D_y \cap D_{gx} \neq \emptyset$. Since D_x , D_{gx} and D_y are all contained in some elements of γ which star-refines β it follows that D_x , D_{gx} and D_y are all contained in one single element $V \in \beta$. Then from $D_x \subset V$ it follows that $D_{gx} = gD_x \subset gV$ and from the choice of the elements of β it follows that $V \cap gV = \emptyset$. But this contradicts with $D_{gx} \subset V$. Hence the right hand side of the inclusion (**) has to be empty and it follows $E_x \cap E_{gx} = \emptyset$. Therefore λ_i satisfies the second requirement for special coverings.

- (3) The proof that λ_i satisfies the third condition for special coverings needs to be broken down into three cases:
 - (a) First we assume towards a contradiction that both U and V are elements of $\lambda_i \setminus \lambda_{i-1}$. Since $U \cap V \neq \emptyset$ it follows that U and V are contained in some $U_x \in \beta$. Since V is not g -invariant it follows that $g \in G \setminus H_i$ and therefore necessarily $U_x \cap gU_x = \emptyset$. Thus $U \cap gV \subset U_x \cap gU_x = \emptyset$ which is a contradiction to the assumption that $U \cap gV \neq \emptyset$. Hence U and V cannot be both elements of $\lambda_i \setminus \lambda_{i-1}$.
 - (b) The next possibility is that $U \in \lambda_i \setminus \lambda_{i-1}$ but $V \in \lambda_{i-1}$. Then $U \subset U_x$ for some $U_x \in \beta$. Further $U_x \cap V \neq \emptyset$ and $U_x \cap gV \neq \emptyset$. Thus $x \in V$ and $x \in gV$ by the choice of U_x and this is a contradiction to the assumption $V \cap gV = \emptyset$. Therefore the second case is not possible, either.
 - (c) The remaining possibility is the case $V \in \lambda_i \setminus \lambda_{i-1}$ but $U \in \lambda_{i-1}$. In a similar fashion as in the previous case we conclude that $U \cap gU \neq \emptyset$. But this time this does not yield a contradiction but it gives the desired result that $U = gU$.
- (4) The fourth condition for special coverings is a straight forward consequence of the requirement (*) which we have put on the choice of the sets D_x . \square

3. Extension to Čech Homology Groups

We are now ready to extend the concept of special homology groups of Section 1 to the Čech homology groups of a compact G -space X where G is a finite cyclic p -group.

Let λ be a special open covering of X . By Proposition 2.36 the action of G on the nerve K_λ is regular and we can apply the results of Section 1 to its chain complex of ordered simplices and its homology groups. The chain complexes and maps of Section 1 depend of course on the choice of the covering λ . We shall indicate this dependency by using lower indices, that is, we will use the notation $\sigma_\lambda, \tau_\lambda, \omega_\lambda, \iota_\lambda$, etc. to label those chain maps.

Next let μ be another special covering of X refining λ and let $p_{\lambda\mu}: \mu \rightarrow \lambda$ be an G -equivariant refinement projection. Then $p_{\lambda\mu}$ induces chain maps $p_{\lambda\mu}: C(K_\mu) \rightarrow C(K_\lambda)$ and $\bar{p}_{\lambda\mu}: C(K_\mu/G) \rightarrow C(K_\lambda/G)$. By simple diagram chasing one shows

Lemma 2.45. *The G -equivariant refinement projection $p_{\lambda\mu}: \mu \rightarrow \lambda$ induces the following chain maps:*

$$\begin{array}{ll} p_{\lambda\mu}: \operatorname{im} \sigma_\mu \rightarrow \operatorname{im} \sigma_\lambda & p_{\lambda\mu}: \ker \sigma_\mu \rightarrow \ker \sigma_\lambda \\ p_{\lambda\mu}: \operatorname{im} \tau_\mu \rightarrow \operatorname{im} \tau_\lambda & p_{\lambda\mu}: \ker \tau_\mu \rightarrow \ker \tau_\lambda \\ \bar{p}_{\lambda\mu}: C(K_\mu/G) \rightarrow C(K_\lambda/G) & \bar{p}_{\lambda\mu}: D(K_\mu) \rightarrow D(K_\lambda) \quad \square \end{array}$$

Similarly we convince ourselves by diagram chasing that all the homomorphisms of the previous section commute with the G -equivariant refinement projection p . That is we get the

Lemma 2.46. *For any special coverings λ and μ of X with μ refining λ and $p_{\lambda\mu}: \mu \rightarrow \lambda$ being a G -equivariant refinement projection we have the following equalities: $p_{\lambda\mu}\sigma_\mu = \sigma_\lambda p_{\lambda\mu}$, $p_{\lambda\mu}\tau_\mu = \tau_\lambda p_{\lambda\mu}$, $p_{\lambda\mu}\omega_\mu = \omega_\lambda p_{\lambda\mu}$, $p_{\lambda\mu}\iota_\mu = \iota_\lambda p_{\lambda\mu}$, $\bar{p}_{\lambda\mu}\theta_\mu = \theta_\lambda \bar{p}_{\lambda\mu}$, etc. \square*

If $q_{\lambda\mu}: \mu \rightarrow \lambda$ is another G -equivariant refinement projection, then the corresponding homomorphisms from Lemma 2.45 are of course different. But yet they induce unique homomorphisms when passing to homology. This will be expressed in the following two results.

Lemma 2.47. *When passing to homology the maps $p_{\lambda\mu*}: H(K_\mu) \rightarrow H(K_\lambda)$ and $\bar{p}_{\lambda\mu*}: H(K_\mu/G) \rightarrow H(K_\lambda/G)$ are independent of the choice of $p_{\lambda\mu}$.*

PROOF. Let $p_{\lambda\mu}, q_{\lambda\mu}: \mu \rightarrow \lambda$ two G -equivariant refinement projections. Then p and q are contiguous maps and thus the induced chain maps $p_{\lambda\mu}, q_{\lambda\mu}: C(K_\mu) \rightarrow C(K_\lambda)$ are chain homotopic maps. Therefore $p_{\lambda\mu*} = q_{\lambda\mu*}: H(K_\mu) \rightarrow H(K_\lambda)$ are the same map.

It is evident that also the induced chain maps $\bar{p}_{\lambda\mu}, \bar{q}_{\lambda\mu}: K_\mu/G \rightarrow K_\lambda/G$ are contiguous maps (since p and q are) and thus it follows by the same reasoning as before that $\bar{p}_{\lambda\mu*} = \bar{q}_{\lambda\mu*}: H(K_\mu/G) \rightarrow H(K_\lambda/G)$. \square

Lemma 2.48. *When passing to homology the maps $p_{\lambda\mu*}: \bar{H}^\sigma(K_\mu) \rightarrow \bar{H}^\sigma(K_\lambda)$, $p_{\lambda\mu*}: H^\sigma(K_\mu) \rightarrow H^\sigma(K_\lambda)$, $p_{\lambda\mu*}: \bar{H}^\tau(K_\mu) \rightarrow \bar{H}^\tau(K_\lambda)$, etc. induced by the chain maps of Lemma 2.45 do not depend on the choice of the refinement projection $p_{\lambda\mu}: \mu \rightarrow \lambda$.*

PROOF. Recall that that a chain homotopy $D: p_{\lambda\mu} \simeq q_{\lambda\mu}$ is given by

$$D(U_0, \dots, U_n) := \sum_{i=0}^n (-1)^i (q_{\lambda\mu}(U_0), \dots, q_{\lambda\mu}(U_i), p_{\lambda\mu}(U_i), \dots, p_{\lambda\mu}(U_n))$$

where (U_0, \dots, U_n) is an ordered simplex of K_μ (see the proof of Lemma 1.6) Since both $p_{\lambda\mu}$ and $q_{\lambda\mu}$ are G -equivariant refinement projections it follows that the chain

homotopy D is G -equivariant, too. Using this G -equivariance of D we conclude that $D(\text{im } \sigma_\mu) \subset \text{im } \sigma_\lambda$, $D(\ker \sigma_\mu) \subset \ker \sigma_\lambda$, $D(\text{im } \tau_\mu) \subset \text{im } \tau_\lambda$ and $D(\ker \tau_\mu) \subset \ker \tau_\lambda$. Thus the restriction $D|_{\text{im } \sigma_\mu}$ of D to $\text{im } \sigma_\mu$ is a chain homotopy

$$D|_{\text{im } \sigma_\mu}: p_{\lambda\mu}|_{\text{im } \sigma_\mu} \simeq q_{\lambda\mu}|_{\text{im } \sigma_\mu},$$

and for similar reason we have that $p_{\lambda\mu}|_{\ker \sigma_\mu}$ and $q_{\lambda\mu}|_{\ker \sigma_\mu}$ are chain homotopic as well as $p_{\lambda\mu}|_{\text{im } \tau_\mu}$ and $q_{\lambda\mu}|_{\text{im } \tau_\mu}$ are chain homotopic and $p_{\lambda\mu}|_{\ker \tau_\mu}$ and $q_{\lambda\mu}|_{\ker \tau_\mu}$ are chain homotopic. Thus passing to homology yields in every case a unique map independent of the choice of the refinement projection.

It remains to show that the restriction of $\bar{p}_{\lambda\mu}$ and $\bar{q}_{\lambda\mu}$ to $D(K_\mu)$ are chain homotopic. From the definition $D(K_\mu) = \pi(\ker \sigma_\mu)$ it follows that $\bar{D}(D(K_\mu)) = \bar{D}(\pi(\ker \sigma_\mu)) = \pi D(\ker \sigma_\mu) \subset \pi(\ker \sigma_\lambda) = D(K_\lambda)$. Thus the restriction $\bar{D}|_{D(K_\mu)}$ of the chain homotopy \bar{D} to $D(K_\mu)$ defines a chain homotopy $\bar{p}_{\lambda\mu}|_{D(K_\mu)} \simeq \bar{q}_{\lambda\mu}|_{D(K_\mu)}$. \square

Because of the previous results and the nature of the refinement projections – that is $p_{\lambda\mu}p_{\mu\eta} = p_{\lambda\eta}$ and $p_{\lambda\lambda} = \text{id}$ for every $\lambda \leq \mu \leq \eta$ in $\text{Cov}_{\text{sp}}(X)$ – we can construct the inverse systems

$$\begin{aligned} \{H_k(K_\lambda), p_{\lambda\mu^*}\}, & \quad \{H_k(K_\lambda/G), \bar{p}_{\lambda\mu^*}\}, & \quad \{H_k^\sigma(K_\lambda), p_{\lambda\mu^*}\}, \\ \{\bar{H}_k^\sigma(K_\lambda), p_{\lambda\mu^*}\}, & \quad \text{etc.} \end{aligned}$$

directed by $\text{Cov}_{\text{sp}}(X)$ using the homology groups introduced in Section 1.

Proposition 2.49. *We have the two isomorphisms*

$$\check{H}_k(X) \cong \varprojlim H_k(K_\lambda) \quad \text{and} \quad \check{H}_k(X/G) \cong \varprojlim H_k(K_\lambda/G)$$

where the limits are taken over the inverse systems as described above.

PROOF. The first isomorphism is a straight forward consequence of Proposition 2.39. To verify the second isomorphism we need to do some more work. The following proof is based on [Bre72, p. 135] and completes the exposition in Yang's article [Yan60, p. 209].

We want to construct an isomorphism

$$\check{H}_k(X/G) \cong \varprojlim H_k(K_\lambda/G).$$

Note that the left hand side of this equation is the limit of an inverse system directed by $\text{Cov}(X/G)$ whereas the limit on the right hand side is directed by the set $\text{Cov}_{\text{sp}}(X)$.

First we want to define an order preserving map

$$\varphi: \text{Cov}_{\text{sp}}(X) \rightarrow \text{Cov}(X/G).$$

Let $\lambda \in \text{Cov}_{\text{sp}}(X)$. Then λ is a self-indexed covering of X . Define an indexed covering λ_1

$$\lambda_1: V \rightarrow \mathcal{P}(X/G)$$

of the orbit space X/G as follows. To simplify notation set $U^* := \{gU : g \in G\}$ for every $U \in \lambda$. We use as a index set for the covering λ_1 the set $V := \{U^* : U \in \lambda\}$. Now define

$$\lambda_1(U^*) := \pi(U),$$

for every $U^* \in V$. Here in this definition π denotes the canonical projection onto the orbit space $\pi: X \rightarrow X/G$. Note that this gives a well defined map since if $U' \in U^*$ is another representative, then $\pi(U') = \pi(U)$. Furthermore λ_1 is an open covering of X/G since π is an open map and λ is an open covering of X . Define $\varphi(\lambda) := \lambda_1$ for every $\lambda \in \text{Cov}_{\text{sp}}(X)$. Then φ is a map $\varphi: \text{Cov}_{\text{sp}}(X) \rightarrow \text{Cov}(X/G)$. It is order preserving because whenever μ is a special covering of X refining λ then apparently $\varphi(\mu) = \mu_1$ refines $\varphi(\lambda) = \lambda_1$.

By the definition of the simplicial orbit space K_λ/G (see Definition 1.16) the set of vertices of K_λ/G are precisely the set of vertices of K_{λ_1} , namely the set V . We claim that the identity on V extends to a simplicial isomorphism $K_{\lambda_1} \rightarrow K_\lambda/G$. That is, we have to show that $\{U_0^*, \dots, U_k^*\}$ is a k -simplex of K_{λ_1} if and only if $\{U_0^*, \dots, U_k^*\}$ is a k -simplex of K_λ/G .

Thus assume first that $\{U_0^*, \dots, U_k^*\}$ is a k -simplex of K_{λ_1} . By definition this means that $\lambda_1(U_0^*) \cap \dots \cap \lambda_1(U_k^*) = \pi(U_0) \cap \dots \cap \pi(U_k) \neq \emptyset$. This can only happen if there exist elements $g_i \in G$ such that $g_0 U_0 \cap \dots \cap g_k U_k \neq \emptyset$ and therefore $\{g_0 U_0 \cap \dots \cap g_k U_k\}$ is a simplex of K_λ . But then $\{U_0^*, \dots, U_k^*\} = \{(g_0 U_0)^*, \dots, (g_k U_k)^*\}$ is a k -simplex of K_λ/G . On the other hand let $\{U_0^*, \dots, U_k^*\}$ be a k -simplex of K_λ/G . Then it is evident that $\{U_0^*, \dots, U_k^*\}$ is also a k -simplex of K_{λ_1} .

Thus we have verified that the identity $\text{id}: V \rightarrow V$ induces a simplicial isomorphism $K_{\lambda_1} \rightarrow K_\lambda/G$. Actually we have even shown more, namely $K_{\lambda_1} = K_\lambda/G$. As a consequence of this we have that $H_k(K_{\lambda_1}) = H_k(K_\lambda/G)$ for every integer k .

It follows easily that if μ is another special covering of X refining λ , then for a given equivariant refinement projection $p: \mu \rightarrow \lambda$ we get a refinement projection $p_1: \mu_1 \rightarrow \lambda_1$ when we define $p_1(U^*) := (p(U))^*$. Then the commutativity of the diagram

$$\begin{array}{ccc} K_{\mu_1} & \xrightarrow{\text{id}} & K_\mu/G \\ p_1 \downarrow & & \downarrow p \\ K_{\lambda_1} & \xrightarrow{\text{id}} & K_\lambda/G \end{array}$$

is evident and thus we have that for any refinement projection $q: \mu_1 \rightarrow \lambda_1$ we get – when passing to homology – the commutative diagram

$$\begin{array}{ccc} H_k(K_{\mu_1}) & \xrightarrow{\text{id}} & H_k(K_\mu/G) \\ q_* \downarrow & & \downarrow p_* \\ H_k(K_{\lambda_1}) & \xrightarrow{\text{id}} & H_k(K_\lambda/G) \end{array}$$

Thus the order preserving map $\varphi: \text{Cov}_{\text{sp}}(X) \rightarrow \text{Cov}(X/G)$ as defined above together with the identities

$$\text{id}: H_k(K_{\varphi(\lambda)}) \rightarrow H_k(K_\lambda/G)$$

for every $\lambda \in \text{Cov}_{\text{sp}}(X)$ defines a map

$$F: \{H_k(K_\lambda), p_*\} \rightarrow \{H_k(K_\lambda/G), p_*\}$$

of inverse systems, directed by $\text{Cov}(X/G)$ and $\text{Cov}_{\text{sp}}(X)$ respectively. Its limit

$$f: \check{H}_k(X/G) \rightarrow \varprojlim H_k(K_\lambda/G).$$

is apparently an isomorphism of groups. □

Note that we use the symbol “ π ” to denote two kind of maps. The first one is the simplicial projection

$$\pi: K_\lambda \rightarrow K_\lambda/G$$

onto the simplicial orbit space, when passing to homology it defines a map of inverse systems

$$\{H_k(K_\lambda), p_*\} \rightarrow \{H_k(K_\lambda/G), p_*\}$$

directed by $\text{Cov}_{\text{sp}}(X)$ and its limit is a homomorphism $\pi_*: \check{H}_k(X) \rightarrow \varprojlim H_k(K_\lambda/G)$. The second map is the continuous projection

$$\pi: X \rightarrow X/G$$

and it induces a homomorphism $\pi_*: \check{H}_k(X) \rightarrow \check{H}_k(X/G)$ in Čech homology. We shall see next that this abuse of notation will not cause problems since if we identify the Čech group $\check{H}_k(X/G)$ with the limit group $\varprojlim H_k(K_\lambda/G)$ by the isomorphism f of the previous proof, then both homomorphisms agree.

Denote by φ_2 the order preserving map

$$\varphi_2: \text{Cov}(X/G) \rightarrow \text{Cov}(X), \lambda_1 \mapsto \pi^{-1}(\lambda')$$

where $\pi^{-1}(\lambda')$ is the open covering

$$\pi^{-1}(\lambda_1): V_{\lambda'} \rightarrow \mathcal{P}(X), v \mapsto \pi^{-1}(\lambda'(v))$$

as it is known from the definition of the induced homomorphism in Čech homology.

Now if λ is a special covering of X , then it follows that $\lambda_2 := \varphi_2(\varphi_1(\lambda))$ is an open covering of X indexed by $V_{\lambda_2} := \{U^* : U \in \lambda\}$. It is given by

$$\lambda_2(U^*) = \pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU$$

Now λ_2 is clearly refined by λ and one possible refinement projection is

$$p: \lambda \rightarrow V_{\lambda_2}, U \mapsto U^*.$$

If $\{U_0, \dots, U_k\}$ is a k -simplex in K_λ , then using the fact that λ satisfies property P_1 it follows that $\{U_0^*, \dots, U_k^*\}$ is a k -simplex of K_{λ_1} . Thus p maps the k -simplex $\{U_0, \dots, U_k\}$ to the k -simplex $\{U_0^*, \dots, U_k^*\}$. Then the inclusion into the complex K_{λ_1} followed by the identity $K_{\lambda_1} = K_\lambda/G$ leaves the k -simplex $\{U_0^*, \dots, U_k^*\}$ unchanged.

On the other hand the canonical simplicial projection $\pi: K_\lambda \rightarrow K_\lambda/G$ onto the simplicial orbit space maps the k -simplex $\{U_0, \dots, U_k\}$ also to the k -simplex $\{U_0^*, \dots, U_k^*\}$ of K_λ/G .

From this observation follows the next result which is not explicitly stated in Yang's article, namely

Proposition 2.50. *If we identify $\check{H}_k(X/G)$ with $\varprojlim H_k(K_\lambda/G)$ via the isomorphism f of the proof of Proposition 2.49, then the canonical projection $\pi: X \rightarrow X/G$ onto the orbit space and the canonical simplicial projection $\pi: K_\lambda \rightarrow K_\lambda/G$ onto the simplicial orbit space define the same homomorphism $\pi_*: \check{H}_k(X) \rightarrow \check{H}_k(X/G)$. \square*

Next we extend the definition of special homology groups of simplicial complexes to the compact G -spaces X .

Definition 2.51 (Special Homology Groups). The k -th special homology groups of the compact G -space X are the following four inverse limits:

$$\begin{aligned} \bar{H}_k^\sigma(X) &:= \varprojlim \bar{H}_k^\sigma(K_\lambda) & H_k^\sigma(X) &:= \varprojlim H_k^\sigma(K_\lambda) \\ \bar{H}_k^\tau(X) &:= \varprojlim \bar{H}_k^\tau(K_\lambda) & H_k^\tau(X) &:= \varprojlim H_k^\tau(K_\lambda) \end{aligned}$$

where the limit is taken over the limiting systems as described before.

Moreover we define $I_k(X)$ to be the limit of the inverse system $\{I_k(K_\lambda), \bar{p}_*\}$.

Proposition 2.52. *Let λ and μ be two special coverings of X where μ refines λ . Let $p_{\lambda\mu}: \mu \rightarrow \lambda$ be a G -equivariant refinement projection. Further denote by*

$$H_\eta: \dots \longrightarrow (H'_\eta)_k \longrightarrow (H_\eta)_k \longrightarrow (H''_\eta)_k \xrightarrow{\partial_*} (H'_\eta)_{k-1} \longrightarrow \dots$$

any of the long exact homology sequences of Proposition 2.22 for $\eta = \lambda, \mu$. Then $p_{\lambda\mu}$ induces an homomorphism of long exact sequences $p_{\lambda\mu}: H_\mu \rightarrow H_\lambda$.*

PROOF. We have to show that every square commutes in the diagram

$$\begin{array}{ccccccccc} \dots & \longrightarrow & (H'_\mu)_k & \longrightarrow & (H_\mu)_k & \longrightarrow & (H''_\mu)_k & \xrightarrow{\partial_*} & (H'_\mu)_{k-1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & (H'_\lambda)_k & \longrightarrow & (H_\lambda)_k & \longrightarrow & (H''_\lambda)_k & \xrightarrow{\partial_*} & (H'_\lambda)_{k-1} & \longrightarrow & \dots \end{array}$$

I II III

where the vertical arrows are induced by the refinement projection $p_{\lambda\mu}$. Now the commutativity of the squares I and II is a direct consequence of Lemma 2.46 and the commutativity of the remaining square III follows from the fact that the connecting homomorphism ∂_* is a natural transformation (see for example [Spa66, p. 182]). \square

In other words each of the long exact homology sequences of Proposition 2.22 together with the homomorphisms induced by the refinement projections gives rise to an inverse system of long exact sequences.

Now homology groups $H_k(K_\lambda)$, $H_k^\sigma(K_\lambda)$, $\overline{H}_k^\sigma(K_\lambda)$, $H_k^\tau(K_\lambda)$, $\overline{H}_k^\tau(K_\lambda)$, $I_k(K_\lambda)$ and $H_k(K_\lambda/G)$ are all compact groups when λ is a special covering (since special coverings are finite and the coefficient groups are compact). Therefore we can pass to the limit (which is under these conditions again a long exact sequence of compact groups, see [ES52, p. 226, Theorem 5.6]) which yields then the following result.

Proposition 2.53. *The following sequences are exact:*

$$\dots \longrightarrow H_k^\sigma(X) \xrightarrow{\omega_*} \check{H}_k(X) \xrightarrow{\sigma_*} \overline{H}_k^\sigma(X) \longrightarrow H_{k-1}^\sigma(X) \xrightarrow{\omega_*} \dots \quad (2.54)$$

$$\dots \longrightarrow H_k^\tau(X) \xrightarrow{\omega'_*} \check{H}_k(X) \xrightarrow{\tau_*} \overline{H}_k^\tau(X) \longrightarrow H_{k-1}^\tau(X) \xrightarrow{\omega'_*} \dots \quad (2.55)$$

$$\dots \longrightarrow \overline{H}_k^\tau(X) \xrightarrow{\omega_* \iota_*} \check{H}_k(X) \xrightarrow{\pi_*} \check{H}_k(X/G) \longrightarrow \overline{H}_{k-1}^\tau(X) \xrightarrow{\omega_* \iota_*} \dots \quad (2.56)$$

$$\dots \longrightarrow \overline{H}_k^\tau(X) \xrightarrow{\iota_*} H_k^\sigma(X) \xrightarrow{\pi_*} I_k(X) \longrightarrow \overline{H}_{k-1}^\tau(X) \xrightarrow{\iota_*} \dots \quad (2.57)$$

and

$$\dots \longrightarrow I_k(X) \xrightarrow{\theta_*} \check{H}_k(X/G) \xrightarrow{\kappa_*} \overline{H}_k^\sigma(X) \longrightarrow I_{k-1}(X) \xrightarrow{\theta_*} \dots \quad (2.58)$$

where the non-labeled homomorphisms are the appropriate connecting homomorphisms. \square

We obtain the following collection of technical results which are in turn the a consequence of the commutativity of diagram (2.7), Proposition 2.22, Corollary 2.32 and Lemma 2.30 from Section 1.

Lemma 2.59. *The diagram*

$$\begin{array}{ccc}
 \check{H}(X) & & \\
 \downarrow \pi_* & \searrow \sigma_* & \\
 \check{H}(X/G) & \xrightarrow{\kappa_*} & \overline{H}^\sigma(X)
 \end{array} \tag{2.60}$$

commutes. □

Lemma 2.61. *The homomorphism $\iota_*: \overline{H}_k^\sigma(X) \rightarrow H_k^r(X)$ is the identity.* □

Lemma 2.62. *Let $e \in \check{H}_k(X/G)$. Then $\pi_* \omega'_* \iota'_* \kappa_*(e) = [r]e$.* □

Lemma 2.63. *For any G -invariant $e \in \check{H}_k(X)$ we have $\omega'_* \iota'_* \sigma_*(e) = [r]e$.* □

4. Special Homology Groups for \mathcal{Z}_p

It is in this section that we will extend the definition of special homology groups to the group of p -adic integers \mathcal{Z}_p acting on a compact space X . The idea is the utilization of the fact that \mathcal{Z}_p is the inverse limit of cyclic p -groups together with the continuity of the Čech homology.

Recall the notation introduced in Section 7 of the previous chapter regarding the group of p -adic integers \mathcal{Z}_p and its subgroups. Since the collection of open subgroups $\{\mathbf{Z}_i\}$ forms a neighborhood system of the neutral element it follows easily that the limit of the inverse system $\{X/\mathbf{Z}_i, \pi_i^j\}$ (where $\pi_i^j: X/\mathbf{Z}_j \rightarrow X/\mathbf{Z}_i$ denote the canonical projections) is homeomorphic to X and that the inverse limit of the canonical projections $\pi_i: X \rightarrow X/\mathbf{Z}_i$ defines a homeomorphism

$$X \cong \varprojlim X/\mathbf{Z}_i$$

Now the projection maps π_i^j induce continuous homomorphisms π_{i*}^j when passing to Čech homology and we get that $\{\check{H}_k(X/\mathbf{Z}_i), \pi_{i*}^j\}$ forms an inverse system of compact groups (recall that we assume the compact group \mathcal{R} as the coefficient groups). It follows then from the continuity of the Čech homology theory (see Theorem 1.41) that the projections $\pi_i: X \rightarrow X/\mathbf{Z}_i$ define an isomorphism

$$\check{H}_k(X) \cong \varprojlim \check{H}_k(X/\mathbf{Z}_i). \tag{2.64}$$

The action of the p -adic integers \mathcal{Z}_p on X induces in the usual way an action of $\mathbb{Z}_{[i]}$ on the orbit space X/\mathbf{Z}_i . Since the $\mathbb{Z}_{[i]}$ are finite cyclic p -groups we can apply the results of the previous section to this collection of spaces. In order to do this we fix an element $\hat{g} \in \mathcal{Z}_p \setminus \mathbf{Z}_1$. We denote $g_i := h_i(\hat{g})$ for each $i \in \mathbb{N}$.

One verifies that g_i is a generator of the cyclic p -group $\mathbb{Z}_{[i]}$ which acts on X/\mathbf{Z}_i . We can apply the results of the previous section. We will emphasize the dependency on $\mathbb{Z}_{[i]}$ of the maps defined this way by adding “ i ” to the index of those maps. That is we have

$$\sigma_{i*} = \text{id} + g_{i*} + g_{i*}^2 + \dots + g_{i*}^{[i]-1} \quad \text{and} \quad \tau_{i*} = \text{id} - g_{i*}$$

and further $\omega_{i*}, \omega'_{i*}, \iota_{i*}, \iota'_{i*}, \pi_{i*}, \theta_{i*}$ and so on.

Now the isomorphism in (2.64) is the key to the extension of the concept of special homology groups to the action of the p -adic integers. Like in the section before we will define special homology groups as limits of appropriate inverse systems, which will now be directed by the natural numbers and with the projections induced by the canonical projections $\pi_i^j: X/\mathbf{Z}_j \rightarrow X/\mathbf{Z}_i$ of the orbit spaces.

But before we begin with the construction of the special homology groups for the p -adic case we shall add the following observation about the use of the symbol “ π ”, which is not made in Yang’s article. The action of \mathcal{Z}_p on X induces

an action of $\mathbb{Z}_{[i]}$ on X/\mathbf{Z}_i . The orbit space $(X/\mathbf{Z}_i)/\mathbb{Z}_{[i]}$ of the latter action is in a natural way homeomorphic to $X/\mathcal{Z}_p = X/\mathbf{Z}_0$. Using this homeomorphism we identify those spaces and the two canonical projection maps

$$\pi_i: X/\mathbf{Z}_i \rightarrow (X/\mathbf{Z}_i)/\mathbb{Z}_{[i]}$$

and

$$\pi_0^i: X/\mathbf{Z}_i \rightarrow X/\mathcal{Z}_p$$

become after this identification essentially the same map. We shall in the following use the symbol X/\mathcal{Z}_p to denote the space $(X/\mathbf{Z}_i)/\mathbb{Z}_{[i]}$, too. When passing to Čech homology we get a homomorphism

$$\pi_{i*} = \pi_{0*}^i: \check{H}(X/\mathbf{Z}_i) \rightarrow \check{H}(X/\mathcal{Z}_p).$$

On the other hand Proposition 2.50 states that the homomorphism π_{i*} is essentially – under suitable identification of $\check{H}(X/\mathcal{Z}_p)$ with $\varprojlim H_k(K_\lambda/\mathbb{Z}_{[i]})$ – identical with the homomorphism $\pi_{i*}: \check{H}(X/\mathbf{Z}_i) \rightarrow \check{H}(X/\mathcal{Z}_p)$ defined by homomorphisms induced by the simplicial projections $\pi_i: K_\lambda \rightarrow K_\lambda/\mathbb{Z}_{[i]}$ when passing to the limit. Thus we can use those different points of view when dealing with this homomorphism.

After this note we shall now begin with the construction process of the special homology groups for the p -adic case.

Lemma 2.65. *Let λ_i be a special covering of X/\mathbf{Z}_i and $\lambda_j := (\pi_i^j)^{-1}(\lambda_i)$ for $i < j$. Then we have the commutative diagrams*

$$\begin{array}{ccc} C(K_{\lambda_j}) & \xrightarrow{\sigma_j} & C(K_{\lambda_j}) \\ \pi_i^j \downarrow & & \downarrow \pi_i^j \\ C(K_{\lambda_i}) & \xrightarrow{[j-i]\sigma_i} & C(K_{\lambda_i}) \end{array} \quad \text{and} \quad \begin{array}{ccc} C(K_{\lambda_j}) & \xrightarrow{\tau_j} & C(K_{\lambda_j}) \\ \pi_i^j \downarrow & & \downarrow \pi_i^j \\ C(K_{\lambda_i}) & \xrightarrow{\tau_i} & C(K_{\lambda_i}) \end{array}$$

PROOF. By construction we have the commutative diagram

$$\begin{array}{ccc} C(K_{\lambda_j}) & \xrightarrow{g_j} & C(K_{\lambda_j}) \\ \pi_i^j \downarrow & & \downarrow \pi_i^j \\ C(K_{\lambda_i}) & \xrightarrow{g_i} & C(K_{\lambda_i}) \end{array}$$

Then we have

$$\pi_i^j \sigma_j = \sum_{k=0}^{[j]-1} \pi_i^j g_j^k = \sum_{k=0}^{[j]-1} g_i^k \pi_i^j = [j-i] \sum_{k=0}^{[i]-1} g_i^k \pi_i^j = [j-i] \sigma_i \pi_i^j$$

(since $g_i^{[i]} = \text{id}$) and

$$\pi_i^j \tau_j = \pi_i^k (1 - g_j) = (1 - g_i) \pi_i^j = \tau_i \pi_i^j \quad \square$$

Passing to homology and taking the limit over all special coverings of X/\mathbf{Z}_i we get instantly

Corollary 2.66. *We have the following commutative diagrams*

$$\begin{array}{ccc} \check{H}(X/\mathbf{Z}_j) & \xrightarrow{\sigma_{j*}} & \check{H}(X/\mathbf{Z}_j) \\ \pi_{i*}^j \downarrow & & \downarrow \pi_{i*}^j \\ \check{H}(X/\mathbf{Z}_i) & \xrightarrow{[j-i]\sigma_{i*}} & \check{H}(X/\mathbf{Z}_i) \end{array} \quad \text{and} \quad \begin{array}{ccc} \check{H}(X/\mathbf{Z}_j) & \xrightarrow{\tau_{j*}} & \check{H}(X/\mathbf{Z}_j) \\ \pi_{i*}^j \downarrow & & \downarrow \pi_{i*}^j \\ \check{H}(X/\mathbf{Z}_i) & \xrightarrow{\tau_{i*}} & \check{H}(X/\mathbf{Z}_i) \end{array}$$

□

From the commutativity of the diagrams in Lemma 2.65 it follows that if $c \in \text{im } \tau_j$ then $\pi_i^j(c) \in \text{im } \tau_i$, and if $c \in \ker \tau_j$ then $\pi_i^j(c) \in \ker \tau_i$. Since $\text{im } \sigma = \ker \tau$ by Lemma 2.17 it follows from the previous statement that if $c \in \text{im } \sigma_j$ then $\pi_i^j(c) \in \text{im } \sigma_i$.

But on the other hand it is straightforward to construct a chain $c \in \ker \sigma_j$ such that $\sigma_i \pi_i^j(c) \neq 0$. To see this let S be an orbit of an arbitrary ordered simplex of K_{λ_j} and let $0 \neq \alpha := 1/[j] \in \mathcal{R}$. Define $c := \alpha \sum_{s \in S} s$. Then

$$\sigma_j(c) = \frac{[j]}{|S|} \alpha_S \sum_{s \in S} s$$

with $\alpha_S = |S| \cdot \alpha = |S|/[j]$ and thus $\sigma_j(c) = 0$, that is $c \in \ker \sigma_j$. But since the action of $\mathbb{Z}_{[i]}$ on K_{λ_i} is by construction the same as the action of $\mathbb{Z}_{[j]}$ on K_{λ_j} we have that

$$\sigma_i \pi_i^j(c) = \frac{[i]}{|S|} \alpha_S \sum_{s \in S} s$$

with the same value of α_S , that is $\alpha = |S|/[j]$. Thus $\sigma_i(c) = 1/[j-i] \sum s$ which is a nontrivial element in $C(K_{\lambda_i})$. Therefore $\pi_i^j(c) \notin \ker \sigma_i$ even though $c \in \ker \sigma_j$ as claimed.¹

Thus the projection $\pi_i^j: X/\mathbf{Z}_j \rightarrow X/\mathbf{Z}_i$ induces the homomorphisms

$$\begin{aligned} \pi_{i*}^j: H^\tau(K_{\lambda_j}) &\rightarrow H^\tau(K_{\lambda_i}), & \pi_{i*}^j: \overline{H}^\tau(K_{\lambda_j}) &\rightarrow \overline{H}^\tau(K_{\lambda_i}), \\ \pi_{i*}^j: \overline{H}^\sigma(K_{\lambda_j}) &\rightarrow \overline{H}^\sigma(K_{\lambda_i}), \end{aligned}$$

but π_i^j does *not* induce a homomorphism

$$\pi_{i*}^j: H^\sigma(K_{\lambda_j}) \rightarrow H^\sigma(K_{\lambda_i}).$$

Now we can form the inverse systems

$$\{\overline{H}^\sigma(X/\mathbf{Z}_i), \pi_*\}, \quad \{\overline{H}^\tau(X/\mathbf{Z}_i), \pi_*\} \quad \text{and} \quad \{H^\tau(X/\mathbf{Z}_i), \pi_*\}$$

directed by the natural numbers \mathbb{N} . But since π_i^j does not induce a homomorphism $H^\sigma(K_{\lambda_j}) \rightarrow H^\sigma(K_{\lambda_i})$ the collection $\{H^\sigma(X/\mathbf{Z}_i), \pi_*\}$ does not form an inverse system. Thus we can only define three (and not four) special homology groups. This leads to the following definition.

¹Note that Yang claims in his paper by mistake – see [Yan60, p. 212] – that π_i^j induces a homomorphism $\pi_{i*}^j: H^\sigma(X/\mathbf{Z}_j) \rightarrow H^\sigma(X/\mathbf{Z}_i)$. Yet he essentially does not use this wrong result.

Definition 2.67 (Special Homology Groups for the p -Adic Integers). Let X be a compact \mathcal{Z}_p -space. Then the k -th special homology groups of X are defined to be the following inverse limits

$$\begin{aligned}\bar{H}_k^\sigma(X) &:= \varprojlim \bar{H}_k^\sigma(X/\mathbf{Z}_i) \\ H_k^\tau(X) &:= \varprojlim H_k^\tau(X/\mathbf{Z}_i) \\ \bar{H}_k^\tau(X) &:= \varprojlim \bar{H}_k^\tau(X/\mathbf{Z}_i)\end{aligned}$$

where the limits are taken over the limiting systems described before.

Note that we can not extend the definition of the group I_k since we can show in a similar way as above that the projection π_i^j does *not* induce a homomorphism $D_k(K_{\lambda_j}) \rightarrow D_k(K_{\lambda_i})$.

Due to Lemma 2.61 we have that $\bar{H}_k^\sigma(X/\mathbf{Z}_i) = H_k^\tau(X/\mathbf{Z}_i)$ for all $i \in \mathbb{N}$ and thus we have the next result.

Lemma 2.68. *The homomorphism $\iota: \bar{H}^\sigma(X) \rightarrow H^\tau(X)$ is the identity.* \square

So essentially we deal only with two distinct special homology groups in the p -adic case, namely $H_k^\tau(X)$ and $\bar{H}_k^\tau(X)$.

Next we shall study which homomorphisms of the last section can be extended to the new environment.

Lemma 2.69. *When passing to the limit the collection of homomorphisms $\{\tau_{i*}\}$, $\{\omega'_{i*}\}$, $\{\omega_{i*}\nu_{i*}\}$ and $\{\pi_{i*}\}$ define the homomorphisms²*

$$\begin{aligned}\tau_*: \check{H}(X) &\rightarrow \bar{H}^\tau(X), & \omega'_*: H^\tau(X) &\rightarrow \check{H}(X), \\ (\omega\nu)_*: \bar{H}^\tau(X) &\rightarrow \check{H}(X), & \pi_*: \check{H}(X) &\rightarrow \check{H}(X/\mathcal{Z}_p).\end{aligned}$$

PROOF. Since $\pi_i^j(\text{im } \tau_j)$ is contained in $\text{im } \tau_i$ it follows from the second diagram of Lemma 2.65 that there exists the commutative diagram

$$\begin{array}{ccc} C(K_{\lambda_j}) & \xrightarrow{\tau_j} & \text{im } \tau_j \\ \pi_i^j \downarrow & & \downarrow \pi_i^j \\ C(K_{\lambda_i}) & \xrightarrow{\tau_i} & \text{im } \tau_i \end{array} \quad \text{and thus} \quad \begin{array}{ccc} \check{H}(X/\mathbf{Z}_j) & \xrightarrow{\tau_{j*}} & \bar{H}^\tau(X/\mathbf{Z}_j) \\ \pi_{i*}^j \downarrow & & \downarrow \pi_{i*}^j \\ \check{H}(X/\mathbf{Z}_i) & \xrightarrow{\tau_{i*}} & \bar{H}^\tau(X/\mathbf{Z}_i) \end{array}$$

from which we then conclude the existence of the first homomorphism, namely τ_* .

The next two homomorphisms follow in a similar way from the equalities $\omega'_i \pi_i^j = \pi_i^j \omega'_j$, $\omega_i \pi_i^j = \pi_i^j \omega_j$ and $\nu_i \pi_i^j = \pi_i^j \nu_j$ and from the inclusions $\pi_i^j(\text{im } \tau_j) \subset \text{im } \tau_i$ and $\pi_i^j(\ker \tau_j) \subset \ker \tau_i$.

Finally the last homomorphism follows from the commutativity of the diagram

$$\begin{array}{ccc} C(K_{\lambda_j}) & \xrightarrow{\pi_j} & C(K_{\lambda_j}/\mathcal{Z}_p) \\ \pi_i^j \downarrow & & \downarrow \pi_i^j \\ C(K_{\lambda_i}) & \xrightarrow{\pi_i} & C(K_{\lambda_i}/\mathcal{Z}_p) \end{array}$$

²Note that Yang claimed by mistake in his article that $\pi_i \pi_i^j \neq \pi_i^j \pi_j$. Therefore he could not extend the definition of π_* to a homomorphism $\pi_*: \check{H}(X) \rightarrow \check{H}(X/\mathcal{Z}_p)$.

because then – passing to homology and then to the limit over all special coverings λ_i of X/\mathbf{Z}_i – we get the commutative diagram

$$\begin{array}{ccc} \check{H}(X/\mathbf{Z}_j) & \xrightarrow{\pi_{j*}} & \check{H}(X/\mathbf{Z}_p) \\ \pi_{i*}^j \downarrow & & \downarrow \pi_{i*}^j \\ \check{H}(X/\mathbf{Z}_i) & \xrightarrow{\pi_{i*}} & \check{H}(X/\mathbf{Z}_p) \end{array}$$

Observe that here $\pi_i^j: C(K_{\lambda_j}/\mathcal{Z}_p) \rightarrow C(K_{\lambda_i}/\mathcal{Z}_p)$ is by construction the identity and therefore $\pi_{i*}^j: \check{H}(X/\mathcal{Z}_p) \rightarrow \check{H}(X/\mathcal{Z}_p)$ is the identity, too. \square

Note that for the remaining homomorphisms the above extension to the p -adic case cannot be made. By Corollary 2.66 we have that $\pi_{i*}^j \sigma_{j*} = [j - i] \sigma_{i*} \pi_{i*}^j$ and thus the π_{i*}^j and σ_{j*} do not commute. For a similar reason π_{i*}^j does not commute with κ_{j*} . Thus we cannot define the homomorphisms σ_* or κ_* for the p -adic case. Moreover we have already noted that the groups $H^\sigma(X)$ and $I(X)$ cannot be defined in the p -adic case and therefore we cannot define any of the homomorphisms

$$\begin{aligned} \omega_*: H^\sigma(X) &\rightarrow \check{H}(X), & \iota_*: \bar{H}^\tau(X) &\rightarrow H^\sigma(X), \\ \theta_*: I(X) &\rightarrow \check{H}(X) \end{aligned}$$

in the p -adic case.

Note also that we have the commutative diagram

$$\begin{array}{ccccc} \bar{H}^\tau(X/\mathbf{Z}_j) & \xrightarrow{\iota_{j*}} & H^\sigma(X/\mathbf{Z}_j) & \xrightarrow{\omega_{j*}} & \check{H}(X/\mathbf{Z}_j) \\ \pi_{i*}^j \downarrow & & & & \downarrow \pi_{i*}^j \\ \bar{H}^\tau(X/\mathbf{Z}_i) & \xrightarrow{\iota_{i*}} & H^\sigma(X/\mathbf{Z}_i) & \xrightarrow{\omega_{i*}} & \check{H}(X/\mathbf{Z}_i) \end{array}$$

but $\pi_i^j: X/\mathbf{Z}_j \rightarrow X/\mathbf{Z}_i$ does not induce a filler $\pi_{i*}^j: \check{H}(X/\mathbf{Z}_j) \rightarrow \check{H}(X/\mathbf{Z}_i)$ as already noted before.

Proposition 2.70. *We have the long exact homology sequences*

$$\dots \longrightarrow H_k^\tau(X) \xrightarrow{\omega_*'} \check{H}_k(X) \xrightarrow{\tau_*} \bar{H}_k^\tau(X) \longrightarrow H_{k-1}^\tau(X) \xrightarrow{\omega_*'} \dots \quad (2.71)$$

$$\dots \longrightarrow \bar{H}_k^\tau(X) \xrightarrow{(\omega\nu)_*} \check{H}_k(X) \xrightarrow{\pi_*} \check{H}_k(X/\mathcal{Z}_p) \longrightarrow \bar{H}_{k-1}^\tau(X) \xrightarrow{(\omega\nu)_*} \dots \quad (2.72)$$

where the unnamed arrows are the obvious connecting homomorphisms.³

PROOF. The exactness of the sequences is verified using the the long exact homology sequences of Proposition 2.53. The proof is carried out in a similar way as the proof of Proposition 2.53 where we used the long exact homology sequences of Proposition 2.22 to verify our claim. \square

³This is Lemma 5.2 and Corollary 5.4 in Yang's paper, though Yang has a slightly weaker result for the second sequence.

5. Applying Homology Dimension

So far we did not consider the homology dimension of the space X and its implications. In this section we shall see what additional information we can obtain about the homomorphism of the previous two sections using the information.

First we consider the case that the group G acting on the compact space X is a cyclic group of order $[r]$ for some $r \in \mathbb{N}$. In the second part of this section we consider then the case where X is a \mathcal{Z}_p -space.

Proposition 2.73. *Let X be a compact G -space with G an cyclic group of order $[r]$. Let $H \subset G$ be the smallest non-trivial subgroup of G and assume that the fixed point set X^H is of homology dimension $\leq n$. Then $I_k(X) = 0$ for all $k > n$.*

Before we turn to the proof of this proposition we shall add the following

Corollary 2.74. *The homomorphisms*

$$\iota_*: \bar{H}_k^\tau(X) \rightarrow H_k^\sigma(X) \quad \text{and} \quad \kappa_*: \check{H}_{k+1}(X/G) \rightarrow \bar{H}_{k+1}^\sigma(X)$$

are monomorphisms for $k \geq n$ and even isomorphisms for $k > n$.

PROOF OF THE COROLLARY. By Proposition 2.53 we have the two exact sequences

$$I_{k+1}(X) \longrightarrow \bar{H}_k^\tau(X) \xrightarrow{\iota_*} H_k^\sigma(X) \longrightarrow I_k(X)$$

and

$$I_{k+1}(X) \longrightarrow \check{H}_{k+1}(X/G) \xrightarrow{\kappa_*} \bar{H}_{k+1}^\sigma(X) \longrightarrow I_k(X)$$

By the previous result we have that $I_{k+1}(X) = 0$ for $k \geq n$ and thus the homomorphisms $\iota_*: \bar{H}_k^\tau(X) \rightarrow H_k^\sigma(X)$ and $\kappa_*: \check{H}_{k+1}(X/G) \rightarrow \bar{H}_{k+1}^\sigma(X)$ are monomorphisms for $k \geq n$. Moreover if $k > n$ then $I_k(X) = 0$ therefore $\iota_*: \bar{H}_k^\tau(X) \rightarrow H_k^\sigma(X)$ and $\kappa_*: \check{H}_{k+1}(X/G) \rightarrow \bar{H}_{k+1}^\sigma(X)$ are isomorphism for $k > n$. \square

Some preparation is needed for the proof of Proposition 2.73. This preparatory work will be done in the next three minor results.

Lemma 2.75. *Let H be a subgroup of G and denote by F the fixed point set X^H . For any special covering λ of X denote by L_λ the fixed point set of K_λ under the action of H . Then we have*

$$\check{H}(F/G; \mathcal{G}) \cong \varprojlim H(L_\lambda/G; \mathcal{G})$$

where the inverse limit on the right side is taken over all special coverings of X and \mathcal{G} is an arbitrary compact abelian coefficient group.

PROOF. Observe that given an open covering λ' of F there exists a special covering λ of X such that

$$\lambda \cap F := \{U \cap F : U \in \lambda \text{ and } U \cap F \neq \emptyset\}$$

refines λ' . Thus the set of all coverings of the form $\lambda \cap F$ with λ being a special covering of X forms a cofinal subset of the collection of all open coverings of F . All these coverings are regular and thus we can apply Proposition 2.49 and get

$$\check{H}(F/G; \mathcal{G}) \cong \varprojlim H(K_{\lambda \cap F}/G; \mathcal{G}).$$

If λ is a special covering of X it follows from the fourth requirement of a special covering that for any H -invariant $U \in \lambda$ we have $U \cap F \neq \emptyset$ and thus $U \cap F \in \lambda \cap F$. On the other hand, if $U \cap F$ is an element of $\lambda \cap F$ then by the second requirement for a special covering it can be deduced that U is a H -invariant element of λ . Thus the assignment $U \mapsto U \cap F$ defines a bijective correspondence between the vertices of L_λ and $K_{\lambda \cap F}$.

Furthermore it follows from the fourth requirement of a special covering that this assignment extends to a simplicial isomorphism $L_\lambda \rightarrow K_{\lambda \cap F}$. From this it follows then that

$$\varprojlim H(K_{\lambda \cap F}/G; \mathcal{G}) \cong \varprojlim H(L_\lambda/G; \mathcal{G})$$

and altogether we get then

$$\check{H}(F/G; \mathcal{G}) \cong \varprojlim H(K_{\lambda \cap F}/G; \mathcal{G}) \cong \varprojlim H(L_\lambda/G; \mathcal{G}). \quad \square$$

Lemma 2.76. *Let H be a subgroup of G , denote by F the fixed point set X^H and assume that $\text{hd } F \leq n$. Let λ be a special covering of X . Then for any $k > n$ there exists a special covering μ of X refining λ such that any G -equivariant refinement projection $p: \mu \rightarrow \lambda$ induces a trivial homomorphism*

$$p_*: H_k(L_\mu/H; \mathcal{G}) \rightarrow H_k(L_\lambda/H; \mathcal{G}),$$

where L_μ and L_λ denote the fixed point sets of K_μ^H and K_λ^H and \mathcal{G} is an arbitrary compact and discrete – thus finite – coefficient group.

In particular this means that any such refinement projection $p: \mu \rightarrow \lambda$ induces a simplicial map

$$p: C_k(L_\mu/G; \mathcal{G}) \rightarrow C_k(L_\lambda/G; \mathcal{G})$$

which maps $Z_k(L_\mu/G; \mathcal{G})$ into $B_k(L_\lambda/G; \mathcal{G})$.

PROOF. Since $\text{hd } F \leq n$ we have that $\text{hd } F/G \leq n$ and therefore $\check{H}_k(F/G) = 0$ for $k > n$. By the previous lemma we know that then also

$$\varprojlim H_k(L_\lambda/G; \mathcal{G}) = 0.$$

If λ is a special covering, then $H_k(L_\lambda/G; \mathcal{G})$ is finitely generated and since the coefficient group \mathcal{G} is assumed to be compact and discrete it follows that $H_k(L_\lambda/G; \mathcal{G})$ is discrete. Thus $\{0\}$ is an open subset of $H_k(L_\lambda/G; \mathcal{G})$. We have

$$p_\lambda(\varprojlim H_k(L_\lambda/G; \mathcal{G})) \subset \{0\}$$

and we can apply Proposition 1.21. It follows that there exists a special covering μ of X refining λ such that the homomorphism $p_*: H_k(L_\mu/G; \mathcal{G}) \rightarrow H_k(L_\lambda/G; \mathcal{G})$ induced by a G -equivariant refinement projection is trivial. \square

Recall the notation introduced in the first section of this chapter (page 28). The subgroups of G were labeled by H_i ($0 \leq i \leq r$) such that we had the descending series of subgroups ordered by strict inclusion

$$G = H_0 \supset H_1 \supset \dots \supset H_{r-1} \supset H_r = 0$$

Given a regular G -complex K the fixed point sets K^{H_i} were denoted by L_i . Similarly we will denote in the following for any regular G -covering λ of X by L_{λ_i} the fixed point set $K_\lambda^{H_i}$.

Recall further that we denoted by Z_k the cyclic subgroup of \mathcal{R} with exactly k elements. Then we have by Lemma 2.19 that

$$D(K_\lambda) = \sum_{k=0}^{r-1} C(L_{\lambda_i}/G; Z_{[r-i]}),$$

that is every element $c \in D(K_\lambda)$ is of the form $c = c_1 + \dots + c_{r-1}$ with $c_i \in C_k(L_{\lambda_i}/G; Z_{[r-i]})$.

Lemma 2.77. *Assume that the fixed point set X^{H_j} is of homology dimension $\leq n$. Let $k > n$ be a fixed integer. Let μ, λ be special coverings of X with μ refining λ . Let $p: \mu \rightarrow \lambda$ be a G -equivariant refinement projection and assume that p induces a trivial homomorphism*

$$p_*: H_k(L_{\mu j}/G; Z_p) \rightarrow H_k(L_{\lambda j}/G; Z_p).$$

Let $0 \leq j \leq r - i$. If then $c \in Z_k(D(K_\mu))$ is an element of the form $c = c_j + \dots + c_{r-1}$ with $c_i \in C_k(L_{\mu j}/G; Z_{[r-i]})$ then there exists an element $c' \in Z_k(D(K_\mu))$ of the form $c' = c'_{j+1} + \dots + c'_{r-1}$ with $c'_i \in C_k(L_{\lambda j}/G; Z_{[r-i]})$ such that $p(c) - c' \in B_k(D(K_\lambda))$.

PROOF. First observe that if $\alpha \in Z_{[i]}$ then $[j]\alpha \in Z_{[i-j]}$ for $j \leq i$ and $[j]\alpha = 0$ for $j \geq i$. Thus if $i \geq j$ we have for any $c_i \in C_k(L_{\mu j}/G; Z_{r-i})$ that $[r-i-1]c_i = 0$ for $i > j$ and $[r-j-1]c_j \in C_k(L_{\mu j}/G; Z_p)$. Then

$$\begin{aligned} [r-j-1]c &= [r-j-1](c_j + \dots + c_{r-1}) \\ &= [r-j-1]c_j + 0 + \dots + 0 \\ &= [r-j-1]c_j \in C_k(L_{\mu j}/G; Z_p) \end{aligned}$$

and thus

$$\begin{aligned} \partial([r-j-1]c_j) &= \partial([r-j-1]c) \\ &= [r-j-1]\partial(c) \\ &= 0 \end{aligned}$$

since by assumption $c \in Z_k(D(K_\mu))$. Thus $[r-j-1]c_j \in Z_k(L_{\mu j}/G; Z_p)$.

By assumption $p_*: H_k(L_{\mu j}/G; Z_p) \rightarrow H_k(L_{\lambda j}/G; Z_p)$ is trivial, that is the map $p: C_k(L_{\mu j}/G; Z_p) \rightarrow C_k(L_{\lambda j}/G; Z_p)$ maps every element of $Z_k(L_{\mu j}/G; Z_p)$ into $B_k(L_{\lambda j}/G; Z_p)$. In particular $p([r-j-1]c_j) \in B_k(L_{\lambda j}/G; Z_p)$. Thus there must exist an $a'_j \in C_{k+1}(L_{\lambda j}/G; Z_p)$ such that

$$\partial a'_j = p([r-j-1]c_j).$$

Then $a_j := a'_j/[r-j-1]$ is an element of $C_k(L_{\lambda j}/G; Z_{[r-j]})$ and we have by construction $p([r-j-1]c_j) = [r-j-1]\partial a_j$.

We define an element $c' := c'_{j+1} + \dots + c'_{r-1} \in D_k(K_\lambda)$ by

$$\begin{aligned} c'_{j+1} &:= p(c_j) - \partial a_j + p(c_{j+1}), \\ c'_i &:= p(c_i) \quad (\text{for } i > j + 1). \end{aligned}$$

Clearly $c' \in Z_k(D(K_\lambda))$ and $c'_i \in C_k(L_{\lambda j}/G; Z_{[r-i-1]})$ for all $i > j + 1$. Since $L_{\lambda j} \subset L_{\lambda(j+1)}$ and $[r-(j+1)](p(c_j) - \partial a_j) = 0$ it follows that $c'_{j+1} \in C_k(L_{\lambda j}/G; Z_{[r-j-1]})$, too. Thus c' has the form required by the lemma and by construction we have that $p(c) - c' = \partial a_j \in B_k(L_{\lambda j}/G; Z_{[r-j-1]}) \subset B_k(D(K_\lambda))$. \square

PROOF OF PROPOSITION 2.73. Let λ_0 be an arbitrary special covering of X . Then by repeated use of Lemma 2.76 we can construct a sequence of special coverings

$$\lambda_r \longrightarrow \lambda_{r-1} \longrightarrow \lambda_{r-2} \longrightarrow \dots \longrightarrow \lambda_2 \longrightarrow \lambda_1 \longrightarrow \lambda_0$$

such that λ_i is refining λ_{i-1} for all $i > 0$ and such that the refinement projection $p_{\lambda_{i-1}\lambda_i}: \lambda_i \rightarrow \lambda_{i-1}$ induces a trivial homomorphism

$$p_{\lambda_{i-1}\lambda_i*}: H_k(L_{\lambda_i(r-i)}/G; Z_p) \rightarrow H_k(L_{\lambda_{i-1}(r-i)}/G; Z_p)$$

for $i > 0$.

Denote for $0 < i < j$ by $p_{ij}: C_k(K_{\lambda_j}/G) \rightarrow C_k(K_{\lambda_i}/G)$ the composite

$$p_{ij} := p_{\lambda_i\lambda_{i+1}} \circ \dots \circ p_{\lambda_{j-1}\lambda_j}.$$

By repeated application of Lemma 2.77 we get that for any $c \in Z_k(D(K_{\lambda_r}))$ the element $p_{ir}(c)$ is homologous to an element $c' = c'_{r-i} + \cdots + c'_{r-1}$ with $c'_j \in Z_k(C(L_{\lambda_{ij}}/G; Z_{[r-j]}))$. In particular $p_{0r}(c) \in Z_k(D(K_{\lambda_0}))$ is homologous to 0.

Thus $p_{0r*}(e) = 0$ for any $e \in I_k(K_{\lambda_r})$ and therefore $I_k(X)$ is necessarily trivial since the arbitrary choice of the special covering λ_0 . \square

Consider a compact G -space X with $\text{hd } X \leq n$ and denote by H the smallest non-trivial subgroup of G . Let $k > n$ be an arbitrary integer. Then the fixed point set X^H is a closed subset of X and thus $\text{hd } X^H \leq n$ by Proposition 1.54. We have then

$$\overline{H}_k^\tau(X) \cong H_k^\sigma(X) \quad \text{and} \quad \check{H}_{k+1}(X/G) \cong \overline{H}_{k+1}^\sigma(X) \quad (2.78)$$

by Corollary 2.74.

Since the homology dimension of X is finite and G is a finite group the homology dimension is preserved when passing to the orbit space (Proposition 1.59). Thus $\text{hd } X/G \leq n$ and as a consequence $\check{H}_{k'}(X/G) = 0$ whenever $k' > n$. In particular $\check{H}_k(X/G)$ and $\check{H}_{k+1}(X/G)$ are both trivial. Therefore $\overline{H}_{k+1}^\sigma(X) = 0$ since it is isomorphic to $\check{H}_{k+1}(X/G)$ as concluded before. Then due to Lemma 2.61 we have that also $H_{k+1}^\tau(X) = \overline{H}_{k+1}^\sigma(X) = 0$.

From Proposition 2.53 we have the two exact sequences

$$\overline{H}_{k+1}^\sigma(X) \longrightarrow H_k^\sigma(X) \longrightarrow \check{H}_k(X)$$

and

$$\overline{H}_{k+1}^\tau(X) \longrightarrow H_k^\tau(X) \longrightarrow \check{H}_k(X)$$

$\check{H}_k(X) = 0$ since $\text{hd } X < k$ and thus it follows from the first sequence that $H_k^\sigma(X) = 0$ because $\overline{H}_{k+1}^\sigma(X) = 0$. Then also $\overline{H}_k^\tau(X) = 0$ due to (2.78). Since $k > n$ was chosen arbitrarily we have that necessarily $\overline{H}_{k+1}^\tau(X) = 0$, too, and thus we have by the second exact sequence that $H_k^\tau(X) = 0$. Therefore also $\overline{H}_k^\sigma(X) = 0$ due to Lemma 2.61. Thus we have shown

Proposition 2.79. *Assume that $\text{hd } X \leq n$. Then the special homology groups $H_k^\sigma(X)$, $\overline{H}_k^\sigma(X)$, $H_k^\tau(X)$ and $\overline{H}_k^\tau(X)$ are all trivial for $k > n$. \square*

Corollary 2.80. *The homomorphisms*

$$\omega_*: H_k^\sigma(X) \rightarrow \check{H}_k(X) \quad \text{and} \quad \omega'_*: H_k^\tau(X) \rightarrow \check{H}_k(X)$$

are monomorphisms for $k \geq n$.

PROOF. By Proposition 2.53 we have the two exact sequences

$$\overline{H}_{k+1}^\sigma(X) \longrightarrow H_k^\sigma(X) \xrightarrow{\omega_*} \check{H}_k(X) \quad (2.81)$$

and

$$\overline{H}_{k+1}^\tau(X) \longrightarrow H_k^\tau(X) \xrightarrow{\omega'_*} \check{H}_k(X) \quad (2.82)$$

Since $\overline{H}_{k+1}^\sigma(X)$ and $\overline{H}_{k+1}^\tau(X)$ are both trivial for $k \geq n$ it follows that both $\omega_*: H_k^\sigma(X) \rightarrow \check{H}_k(X)$ and $\omega'_*: H_k^\tau(X) \rightarrow \check{H}_k(X)$ are monomorphisms for $k \geq n$. \square

Note that in the above situation the ω_* and ω'_* are isomorphisms of trivial groups for $k > n$.

Finally we shall return to the case where X is a compact \mathcal{Z}_p -space and we will prove a similar result to Proposition 2.73. The next two lemmas – especially the latter one – complete Yang's presentation of Lemma 5.3 in his paper and which will be in this text Proposition 2.85 and the corollary following it.

Lemma 2.83. *For any $0 \leq l \leq i$ the canonical projection $\pi_i: X \rightarrow X/\mathbf{Z}_i$ maps the fixed point set $X^{\mathbf{Z}^j}$ homeomorphically onto $\pi_i(X^{\mathbf{Z}^j})$.*

PROOF. The restriction of π_i to $X^{\mathbf{Z}^i}$ is an injection. The set $X^{\mathbf{Z}^i}$ is compact since it is a closed subset of the compact space X . Since \mathbf{Z}_i is a compact group and X is assumed to be Hausdorff it follows that X/\mathbf{Z}_i is Hausdorff, too. Thus the restriction $\pi_i: X^{\mathbf{Z}^i} \rightarrow X/\mathbf{Z}_i$ is an embedding. \square

Lemma 2.84. *Let $0 \leq j < i$. Then $\pi_i(X)^{\mathbf{Z}^j} = \pi_i(X^{\mathbf{Z}^j})$.*

PROOF. The inclusion $\pi_i(X^{\mathbf{Z}^j}) \subset \pi_i(X)^{\mathbf{Z}^j}$ is apparent. Thus we assume that $x \in X$ is an element such that $g\pi_i(x) = \pi_i(x)$ for all $g \in \mathbf{Z}_j$. Then $\pi_i(gx) = \pi_i(x)$ for all $g \in \mathbf{Z}_j$. Let $g \in \mathbf{Z}_j \setminus \mathbf{Z}_{j+1}$. Then from $\pi_i(gx) = \pi_i(x)$ it follows that there exists an $g' \in \mathbf{Z}_i$ such that $gg'x = x$. Thus $gg' \in G_x$ where G_x denotes the isotropy group of x . Since G_x is closed it must be of the form $G_x = \mathbf{Z}_{j'}$ for some $j' \in \mathbb{N}$. Now since $j < i$ it follows that $h_{j+1}(g) \neq 0$ but $h_{j+1}(g') = 0$ and thus we have $h_{j+1}(gg') \neq 0$. Thus $gg' \in \mathbf{Z}_j$ and therefore $\mathbf{Z}_j \subset G_x$. Hence $j \geq j'$ and thus finally $x \in X^{\mathbf{Z}^j}$. \square

Let $i \geq 1$ be an arbitrary positive integer and assume that the homology dimension of $X^{\mathbf{Z}^i}$ does not exceed n . Then according to the previous two lemmas $\pi_i: X \rightarrow X/\mathbf{Z}_i$ maps the fixed pointset $X^{\mathbf{Z}^{i-1}}$ homeomorphically onto the set $(X/\mathbf{Z}_i)^{\mathbf{Z}^{i-1}}$. Now observe that the action of \mathbf{Z}_{i-1} on X/\mathbf{Z}_i is identical to the action of $H \subset \mathbb{Z}_{[i]}$ on X/\mathbf{Z}_i where H is the smallest non-trivial subgroup of $\mathbb{Z}_{[i]}$. Then $\text{hd}(X/\mathbf{Z}_i)^H \leq n$ and we can apply Proposition 2.73 and conclude that $I_k(X/\mathbf{Z}_i) = 0$.

Now if there exists a n such that $\text{hd} X^{\mathbf{Z}^i} \leq n$ for every $i \in \mathbb{N}$ then we get the following two results which correspond to Lemma 5.3 in Yangs article

Proposition 2.85. *Let X be a compact \mathcal{Z}_p -space. Assume that $\text{hd} X^{\mathbf{Z}^i} \leq n$ for all $i \in \mathbb{N}$. Then we have for every $i > 0$ that $I_k(X/\mathbf{Z}_i) = 0$ whenever $k > n$.* \square

Applying Corollary 2.74 to this result yields then the following result.

Corollary 2.86. *For all $i \in \mathbb{N}$ the homomorphisms*

$$v_{i*}: \bar{H}_k^\tau(X/\mathbf{Z}_i) \rightarrow H_k^\sigma(X/\mathbf{Z}_i) \quad \text{and} \quad \kappa_{i*}: \check{H}_{k+1}(X/\mathcal{Z}_p) \rightarrow \bar{H}_{k+1}^\sigma(X/\mathbf{Z}_i)$$

are monomorphisms for $k \geq n$ and even isomorphisms for $k > n$. \square

6. Summary

Assume that the p -adic integers \mathcal{Z}_p act as a transformation group on a topological space X . Then for every $i \in \mathbb{N}$ the action of \mathcal{Z}_p on X induces in a natural way an action of the cyclic p -group $\mathbb{Z}_{[i]}$ on both the fixed point set $X^{\mathbf{Z}^i}$ and the orbit space X/\mathbf{Z}_i . Therefore not only the results of Section 4 about the action of \mathcal{Z}_p and their improvements in Section 5 are of importance but also the results from Section 3 on the action of cyclic p -groups and their improvements in Section 5 are essential.

In the first section of this chapter we constructed a slightly more general form of Smith theory for simplicial complexes and the action of cyclic p -groups. The major result of Section 3 is that we have no problem to extend the modified Smith theory developed so far to the Čech homology theory. The main key to this result is the existence of special coverings which form a cofinal subset of $\text{Cov}(X)$ in the case that X is a compact topological space.

Therefore we obtain for the action of a cyclic p -group G on X – see Proposition 2.53 – a collection of long exact homology sequences, namely

$$\dots \longrightarrow H_k^\sigma(X) \xrightarrow{\omega_*} \check{H}_k(X) \xrightarrow{\sigma_*} \overline{H}_k^\sigma(X) \longrightarrow H_{k-1}^\sigma(X) \xrightarrow{\omega_*} \dots \quad (2.87)$$

$$\dots \longrightarrow H_k^\tau(X) \xrightarrow{\omega'_*} \check{H}_k(X) \xrightarrow{\tau_*} \overline{H}_k^\tau(X) \longrightarrow H_{k-1}^\tau(X) \xrightarrow{\omega'_*} \dots \quad (2.88)$$

$$\dots \longrightarrow \overline{H}_k^\tau(X) \xrightarrow{\omega_* \iota_*} \check{H}_k(X) \xrightarrow{\pi_*} \check{H}_k(X/G) \longrightarrow \overline{H}_{k-1}^\tau(X) \xrightarrow{\omega_* \iota_*} \dots \quad (2.89)$$

$$\dots \longrightarrow \overline{H}_k^\tau(X) \xrightarrow{\iota_*} H_k^\sigma(X) \xrightarrow{\pi_*} I_k(X) \longrightarrow \overline{H}_{k-1}^\tau(X) \xrightarrow{\iota_*} \dots \quad (2.90)$$

and

$$\dots \longrightarrow I_k(X) \xrightarrow{\theta_*} \check{H}_k(X/G) \xrightarrow{\kappa_*} \overline{H}_k^\sigma(X) \longrightarrow I_{k-1}(X) \xrightarrow{\theta_*} \dots \quad (2.91)$$

Here the special homology groups are with respect to the cyclic p -group G , see Definition 2.51. Furthermore we obtain a collection of small technical but useful results, namely Lemma 2.59 to Lemma 2.63.

Using the homology dimension of the space X we can show that all special homology groups are trivial in dimensions exceeding $\text{hd } X$, which is consistent with $\check{H}_k(X) = 0$ for $k > \text{hd } X$ (Proposition 2.79). As a useful consequence it follows that both homomorphism $\omega: H_k^\sigma(X) \rightarrow \check{H}_k(X)$ and $\omega': H_k^\tau(X) \rightarrow \check{H}_k(X)$ are monomorphism for $k = \text{hd } X$ and trivial isomorphisms for $k > \text{hd } X$ (see Corollary 2.80).

These are the essential results derived for the Smith theory developed for the action of cyclic p -groups and Čech homology theory. Now the extension of this construction process to the action of p -adic integers on compact spaces and Čech homology theory turns out to be more difficult and it is impossible to extend all concepts and results of the previous case to the new setting. The main reason for this is that there is no way to define the special homology groups $H_k^\sigma(X)$ and the homomorphisms $\sigma_*: \check{H}_k(X) \rightarrow \overline{H}_k(X)$ (see the comments preceding Definition 2.67 and Proposition 2.70).

But still we can define the special homology groups $H_k^\tau(X)$ (which are identical to $\overline{H}_k^\sigma(X)$ by Lemma 2.68) and $\overline{H}_k^\tau(X)$ in the p -adic case and we can extend – see Lemma 2.69 – the definition of the four homomorphisms

$$\begin{aligned} \tau_*: \check{H}(X) &\rightarrow \overline{H}^\tau(X), & \omega'_*: H^\tau(X) &\rightarrow \check{H}(X), \\ (\omega \iota)_*: \overline{H}^\tau(X) &\rightarrow \check{H}(X), & \pi_*: \check{H}(X) &\rightarrow \check{H}(X/\mathcal{Z}_p) \end{aligned}$$

to the p -adic case. Thus only two of the long exact sequences of Proposition 2.53 can be safely transferred into the new environment, namely

$$\dots \longrightarrow H_k^\tau(X) \xrightarrow{\omega'_*} \check{H}_k(X) \xrightarrow{\tau_*} \overline{H}_k^\tau(X) \longrightarrow H_{k-1}^\tau(X) \xrightarrow{\omega'_*} \dots \quad (2.92)$$

and

$$\dots \longrightarrow \overline{H}_k^\tau(X) \xrightarrow{(\omega \iota)_*} \check{H}_k(X) \xrightarrow{\pi_*} \check{H}_k(X/\mathcal{Z}_p) \longrightarrow \overline{H}_{k-1}^\tau(X) \xrightarrow{(\omega \iota)_*} \dots \quad (2.93)$$

(see Proposition 2.70). Note that Yang claims by mistake that it would not be possible to extend the definition of the homomorphism $\pi: \check{H}(X) \rightarrow \check{H}(X/G)$ to the p -adic setting (compare Lemma 2.69 with [Yan60, p. 212]). Therefore Yang could not prove the exactness of the sequence (2.93) in its full generality, but just up to

$$\dots \xrightarrow{\pi_*} \check{H}_{n+2}(X/\mathcal{Z}_p) \longrightarrow \overline{H}_{n+1}^\tau(X) \xrightarrow{(\omega \iota)_*} \check{H}_{n+1}(X) \xrightarrow{\pi_*} \check{H}_{n+1}(X/\mathcal{Z}_p)$$

under the assumption that $\text{hd } X^{\mathbf{Z}^i} \leq n$ for every $i \in \mathbb{N}$ (compare with Corollary 5.4 in [Yan60, pp. 212ff.]).

The last important result for the p -adic case is that we can – if the homology dimension of the fixed point sets $X^{\mathbf{Z}_i}$ for all $i \in \mathbb{N}$ is bounded – improve our knowledge about the groups $I_k(X/\mathbf{Z}_i)$ and the homomorphisms

$$v_{i*}: \overline{H}_k^\tau(X/\mathbf{Z}_i) \rightarrow H_k^\sigma(X/\mathbf{Z}_i) \quad \text{and} \quad \kappa_{i*}: \check{H}_{k+1}(X/\mathbf{Z}_p) \rightarrow \overline{H}_{k+1}^\sigma(X/\mathbf{Z}_i)$$

where the special homology groups and the groups $I_k(X/\mathbf{Z}_i)$ are defined for the action of the cyclic p -group $\mathbf{Z}_{[i]}$ on X/\mathbf{Z}_i (see Proposition 2.85 and Corollary 2.86).

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