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The Künneth Formula in Abelian Categories

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Tiivistelmä — Referat — Abstract

In algebraic homology the Künneth Formula and its implications are well known. Given an additive functor t of two variables defined for modules and modules as values, there is a canonical way to extend this functor to chain complexes of modules. The Künneth Formula then states under certain conditions on chain complexes and the underlying ring that a short exact sequence of homology groups exists. More precisely, the formula makes it possible to express the homology groups of $t(C_1, C_2)$ in terms of $H(C_1)$ and $H(C_2)$ using the functor t and its first derived functor t_1.

Although the Künneth Formula is introduced in the literature using modules, it is not a specific property of the category of left (or right) R-modules due to which the formula is valid. The formula proves to be valid in a more general environment: in abelian categories. Just a few additional assumptions (such as the existence of coproducts, projectives and/or injectives) have to be made.

The aim of this master's thesis is to develop the framework of abstract homological algebra in abelian categories and to reveal the reason why the Künneth Formula is valid in this environment.

Resolutions (projective or injective) will be used to define the derived functors t_i (and t'_i) of a given additive functor t, and the existence of certain long exact sequences will be one ingredient of the Künneth Formula. Moreover, under certain conditions there exists a natural homomorphism $\alpha\colon t(H(X_1), H(X_2)) \to Ht(X_1, X_2)$ (or $\alpha'\colon Ht(X_1, X_2) \to t(H(X_1), H(X_2))$), which has certain interesting properties used in the proof of the Künneth Formula.

As an application of the general formulation of the theorem it is shown, how the well-known Künneth Relations for the tensor product $\otimes$ and the homomorphism functor Hom are derived.

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To

my parents,
and Erik and Roman

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0 Introduction

The Künneth Formula and its implications are well known in algebraic homology. Roughly speaking, given an additive functor t which takes two modules as parameters and maps them to an other module, the Künneth Formula establishes under certain constraints a connection between the homology groups $H(C_1)$, $H(C_2)$ and $Ht(C_1, C_2)$ where C_1 and C_2 are chain complexes of modules, namely the existence of a short exact sequence of the form

$$0 \longrightarrow t(H(C_1), H(C_2) \longrightarrow Ht(C_1, C_2) \longrightarrow t_1(H(C_1), H(C_2)) \longrightarrow 0$$

with t_1 being the first derived functor of t. In some cases this sequence is split.

0.1 The Künneth Formula in the Literature

In the literature (see for example [Osb00], [Dol72] or [Vic94]) the Künneth Formula is often only applied to the tensor product $\otimes$ and to the homomorphism functor Hom. Both functors take R-modules as parameters and have R-modules or abelian groups as values. Sometimes there are even some constraints put on the ring R right from the beginning of the development of the Künneth Formula, for example R being commutative or even a principal ideal domain. [Spa66]

The environment of modules and any additional constraints are accompanied with simplifications. For example diagram chasing is possible within the category of left or right modules, left and right modules are equivalent if the underlying ring is commutative, a submodule of a free module is free if the underlying ring is a principal ideal domain). Due to these simplifications deriving the desired result gets easier but the simplifications have also drawbacks: they hide the reason for *why* the Künneth Formula exists.

A slightly different approach to this topic is shown in [CE56], which reveals more of the nature of the Künneth Formula: here first additive functors that still have modules as parameters and values are studied in a more general way. The two main ingredients for the Künneth Formula a certain long exact sequence involving the satellites of the additive functor and a natural homomorphism α with certain interesting properties are derived without specializing to a specific functor.

The approach in [CE56] gives rise to the question: the authors use the categories of left or right modules over given rings, but does the result really rely on the fact that the objects of the categories used are indeed modules?

0.2 Abstract Homology — Living Without Elements

To be able to define the concept of homology the minimum requirement on a category is to be abelian. Actually much of the theory of modules over rings is indeed arrow-theoretic. [Lan99, p. 133]

When doing homology in abelian categories one loses the concept of elements of an object and instead one has to use homomorphisms to formulate and prove results. A proof, which in the category of left (or right) R-modules is done by chasing elements does not necessarily work in a general abelian category. To some extent there are ways around this: one is introduced in [ML71] and is the concept of members of an object (see Appendix A and also [ML71, pp. 200–204]). Two more can can be found in [Osb00, pp. 359–361]. But these concepts work only to a certain degree and cannot help in every situation.

Another implication is that subobjects cannot be defined anymore in the unique way as can be done when having elements. For example the definition of *the* kernel of a module homomorphism is unique whereas the general definition of *a* kernel of a homomorphism f in an abelian category (or any category with a zero object) is a monomorphism k such that $f \circ k = 0$ and through which every homomorphism g such that $f \circ g = 0$ factors uniquely. [Mit65, p. 14] A kernel is therefore only defined up to isomorphism, and for similar reasons other subobjects cannot be defined more precise than up to isomorphism, too. Hence one has in general to be careful when speaking of certain subobjects and drawing conclusions from them. Conversely, when speaking loosely of certain subobjects one has to be aware what is actually going on below the surface.

Nonetheless much (but not all) of what is known from modules over rings is valid in abelian categories as well.

Now in [CE56] the requirements for the Künneth Formula are of categorical nature: the additive functor has to fulfill a certain exactness preserving property, two homomorphisms have to be isomorphisms and certain objects have to be the zero object. This suggests the question whether the Künneth Formula can be derived for abelian categories in general and what additional assumptions have to be made about the categories used.

This master's thesis will give a positive answer to this question: yes, the Künneth Formula can be proven in abelian categories and the amount of additional requirements on the categories is rather small.

The tensor product $\otimes$ and the homomorphism functor Hom will serve as examples how the in the literature known results follow from the general Künneth Formula in abelian categories.

0.3 Symmetries

The Künneth Formula can be applied to left or right exact functors and the variables of interest can be covariant or contravariant. Both properties are dual to each other and hence it is not surprising that the results are not essentially different with respect to those properties.

Studying the Künneth Formula in this more abstract way reveals (as is also done in [CE56]) these symmetries in the formula. For example the direction of certain arrows in diagrams are reversed, left derived functors are used when dealing with right exact functors, right derived functors when dealing with left exact functors and the functors Z and Z' as well as B and B' change their role if the exactness or variance changes. In this thesis I try to emphasize the virtue of these symmetries.

0.4 Deriving the Künneth Formula

The approach used in this thesis to derive the Künneth Formula is the following:

In section 1 the objects of our interest will be defined. On one hand there are additive functors (functors which preserve the additive structure of the homomorphism groups of abelian categories). The additive functors will be classified according to the way they preserve the exactness of short exact sequences. Of importance will be left exact functors and right exact functors. On the other hand, n-graded complexes and their homology complexes will play an important role throughout this thesis (where n will be either equal to 1 or equal to 2, that is chain complexes or bi-graded complexes).

Section 2 will work towards the definition of derived functors. Projective and injective resolutions of objects are introduced. Applying an additive functor to these resolutions will yield a chain complex with a homology complex independent of the choice of those resolutions. The i-th homology object will then yield the definition of the i-th derived functor. Resolving short exact sequences of objects will result in certain long exact sequences involving the derived functors. These sequences will find way into the proof of the Künneth Formula.

The homomorphisms α and α', which — roughly speaking — establish a commutativity relation between an additive functor t and the homology functor H, will be introduced in section 3. These homomorphisms possess interesting properties and they will then enter as the other important ingredient into the proof of the Künneth Formula. This formula will then be derived at the end of the section.

The two special cases of the Künneth Formula are then finally derived in section 4 as an application of the general theorem. As a corollary the Universal Coefficient Theorem will emerge.

In general, this master's thesis follows loosely the way Cartan and Eilenberg derived the Künneth Formula in [CE56], but adapts to the more general environment of abelian categories. Nonetheless there are a few noticeable deviations from [CE56], one of which is that in this thesis derived functors will be used instead of satellites. Although different in concept, satellites and derived functors behave in the given framework alike and yield the same result. [CE56, pp. 90–91] Another but less important difference to [CE56] is that in this thesis the differential operators are of degree -1, which is more commonly used in the literature than degree $+1$. This will affect the sign of the degree of homomorphisms in certain results.

0.5 Preliminaries and Agreements

In this master's thesis I presume that the reader is familiar with basic algebraic concepts. Especially basic knowledge of category theory and the virtue of arguing with morphisms in abelian categories is assumed. A good introduction of these concepts is given by [Mit65] and [ML71].

In particular recall that any abelian category has kernels, cokernels, finite products and finite coproducts, pushouts and pullbacks and is normal and conormal. [Mit65, p. 33]

The categorical notation and formulation in this thesis is mainly influenced by [Mit65]. In particular I use the formulation that a category "has certain objects" instead of "has enough of certain objects". Nonetheless I use deviating from [Mit65] the more modern terms for inverse and direct limits, which are limit and colimit respectively [Osb00, p. 257], but these terms appear in this thesis only in appendix C.

A ring R is always assumed to have unit. The symbol $\mathbb{Z}$ denotes the ring of integers and the symbol $\mathbb{N}$ denotes the set of non-negative integers (this is $0 \in \mathbb{N}$).

If R is a ring, then R–**Mod** denotes the category of left R-modules and **Mod**–R denotes the category of right R-modules. The category of abelian groups is denoted by **AG**.

The composition of two homomorphisms f and g is denoted by fg or $f \circ g$. The zero homomorphism is denoted by 0.

1 Additive Functors and Complexes

In abelian categories the set of morphisms $\mathrm{Mor}(A_1, A_2)$ have the structure of an additively written abelian group. To emphasize this additional structure they are called homomorphisms and the group $\mathrm{Mor}(A_1, A_2)$ is labeled $\mathrm{Hom}(A_1, A_2)$.

In this thesis the interest lies in the functors with abelian categories as domain and codomain which preserve the additive structure of the homomorphism groups $\mathrm{Hom}(A_1, A_2)$. In this section we collect the necessary definitions of additive functors and classify them by their exactness preserving properties.

Moreover abelian categories have all the structure needed to define single-graded and more general n-graded complexes as well as their homology. In this section we define these structures in a general way before we study the effect of additive functors on the homology of those complexes later in this thesis.

1.1 Additive Functors

Assume that $\mathcal{A}$ and $\mathcal{A}'$ are abelian categories and let $t\colon \mathcal{A} \to \mathcal{A}'$ be a functor. The functor t preserves the additive structure of the homomorphism sets, if for any homomorphisms $\varphi, \psi\colon A_1 \to A_2$ of $\mathcal{A}$

$$t(\varphi + \psi) = t(\varphi) + t(\psi)$$

holds. Such a functor will be called *additive*.

In the more general case, where the functor t may have more than one variable, the definition of an additive functor reads

1.1 Definition. *Let $t\colon \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \to \mathcal{A}$ a functor of abelian categories. The functor t is* additive *if for all objects A_k, A_k' of $\mathcal{A}_k$ $(1 \le k \le n)$ and all $\varphi_k, \psi_k \in \mathrm{Hom}(A_k, A_k')$ we have*

$$t(\mathrm{id}, \ldots, \varphi_k + \psi_k, \ldots, \mathrm{id}) = t(\mathrm{id}, \ldots, \varphi_k, \ldots, \mathrm{id}) + t(\mathrm{id}, \ldots, \psi_k, \ldots, \mathrm{id}) \quad (1.2)$$

From this definition it follows that $t(\varphi_1, \ldots, \varphi_n) = 0$ if any of the φ_k equals 0. Moreover, if $A_k = 0$ for some k then also $t(\mathrm{id}_{A_1}, \ldots, \mathrm{id}_{A_n}) = 0$ (since $\mathrm{id}_{A_k} = 0$).

In the following some properties of additive functors will be studied. In order to simplify notation I will only consider a functor $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ of two variables, covariant in the first and contravariant in the second variable. This functor will serve as a prototype for any additive functors of arbitrarily many variables of any variance.

Since abelian categories have images and kernels, exactness can be defined in the usual way:

1.3 Definition. *A sequence of objects*

$$A' \xrightarrow{\varphi} A \xrightarrow{\psi} A''$$

is said to be exact at A *if* $\ker(\psi) = \operatorname{im}(\varphi)$. *A long sequence*

$$\ldots \longrightarrow A_{i+1} \longrightarrow A_i \longrightarrow A_{i-1} \longrightarrow \ldots$$

is said to be exact *if the sequence is exact at* A_i *for all* $i \in \mathbb{Z}$.

One can now classify additive functors by whether they preserve exactness or not, which leads to the

1.4 Definition. *Let* $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ *be an additive functor, covariant in the first and contravariant in the second variable. If for arbitrary exact sequences* $A'_k \to A_k \to A''_k$ *in* $\mathcal{A}_k$ *the sequences*

$$t(A'_1, A_2) \longrightarrow t(A_1, A_2) \longrightarrow t(A''_1, A_2)$$

and

$$t(A_1, A''_2) \longrightarrow t(A_1, A_2) \longrightarrow t(A_1, A''_2)$$

are exact, then the functor t *is said to be* exact.

1.5 Lemma. An additive functor $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ of abelian categories (covariant in the first and contravariant in the other variable) is exact if and only if for all short exact sequences $0 \to A'_k \to A_k \to A''_k \to 0$ in $\mathcal{A}_k$ the sequences

$$0 \longrightarrow t(A'_1, A_2) \longrightarrow t(A_1, A_2) \longrightarrow t(A''_1, A_2) \longrightarrow 0$$

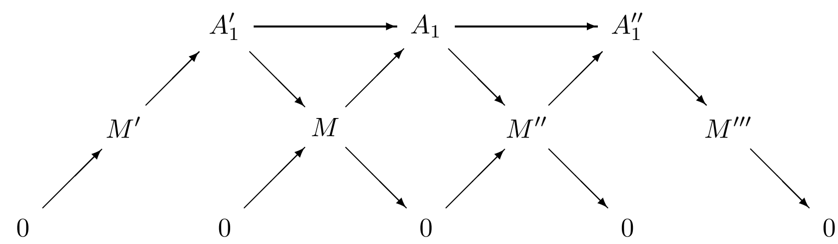
and

$$0 \longrightarrow t(A_1, A''_2) \longrightarrow t(A_1, A_2) \longrightarrow t(A_1, A'_2) \longrightarrow 0$$

are exact.

Proof. "$\Rightarrow$": Clear.

"$\Leftarrow$": Consider the commutative diagram



where $M' := \ker(A'_1 \to A_1)$, $M := \operatorname{im}(A'_1 \to A_1) = \ker(A_1 \to A''_1)$ (exactness of $A'_1 \to A_1 \to A''_1$), $M'' := \operatorname{im}(A_1 \to A''_1)$ and $M''' := \operatorname{coker}(A_1 \to A''_1)$.

Then in addition to the exact sequence $A'_1 \to A_1 \to A''_1$ we have the short exact sequences $0 \to M' \to A'_1 \to M \to 0$, $0 \to M \to A_1 \to M'' \to 0$ and $0 \to M'' \to A''_1 \to M''' \to 0$.

Applying $t(\bullet, A_2)$ to the above diagram results in the short exact sequences $0 \to t(M', A_2) \to t(A'_1, A_2) \to t(M, A_2) \to 0$, $0 \to t(M, A_2) \to t(A_1, A_2) \to t(M'', A_2) \to 0$ and $0 \to t(M'', A_2) \to t(A''_1, A_2) \to t(M''', A_2) \to 0$.

Now because of the commutativity of the triangle A'_1, M, A_1 and because the sequence $t(A'_1, A_2) \to t(M', A_2) \to 0$ is exact, $\operatorname{im}(t(A'_1, A_2) \to t(A_1, A_2)) =$

$\operatorname{im}(t(M, A_2) \to t(A_1, A_2))$. Similarly, due to the commutativity of the triangle A_1, M'', A_1'' and because $t(A_1, A_2) \to t(M'', A_2) \to 0$ is exact, $\ker(t(A_1, A_2) \to t(A_1'', A_2)) = \ker(t(A_1, A_2) \to t(M'', A_2))$.

Since $\operatorname{im}(t(M, A_2) \to t(A_1, A_2)) = \ker(t(A_1, A_2) \to t(M'', A_2))$ it follows that $\operatorname{im}(t(A_1', A_2) \to t(A_1, A_2)) = \ker(t(A_1, A_2) \to t(A_1'', A_2))$ as well, that is the sequence $t(A_1', A_2) \to t(A_1, A_2) \to t(A_1'', A_2)$ is exact.

The proof with respect to the contravariant variable is similar. □

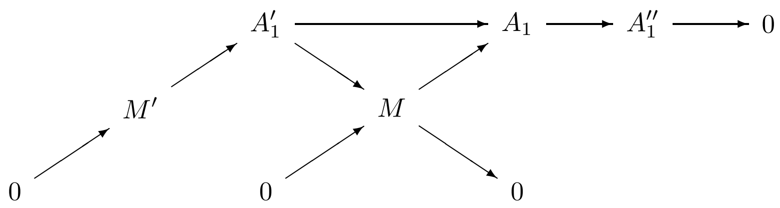
Usually additive functors are not exact, but they may preserve exactness partially which leads to the definition of half exact, left exact and right exact functors:

1.6 DEFINITION. *Let $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be an additive functor of abelian categories, covariant in the first variable and contravariant in the second variable. If for every short exact sequence $0 \to A_k' \to A_k \to A_k'' \to 0$ in $\mathcal{A}_k$,*

1. *the sequences*
$$t(A_1', A_2) \longrightarrow t(A_1, A_2) \longrightarrow t(A_1'', A_2)$$
and
$$t(A_1, A_2'') \longrightarrow t(A_1, A_2) \longrightarrow t(A_1, A_2')$$
are exact, then the functor t is said to be a half exact *functor;*
2. *the sequences*
$$0 \longrightarrow t(A_1', A_2) \longrightarrow t(A_1, A_2) \longrightarrow t(A_1'', A_2)$$
and
$$0 \longrightarrow t(A_1, A_2'') \longrightarrow t(A_1, A_2) \longrightarrow t(A_1, A_2')$$
are exact, then the functor t is said to be a left exact *functor;*
3. *if the sequences*
$$t(A_1', A_2) \longrightarrow t(A_1, A_2) \longrightarrow t(A_1'', A_2) \longrightarrow 0$$
and
$$t(A_1, A_2'') \longrightarrow t(A_1, A_2) \longrightarrow t(A_1, A_2') \longrightarrow 0$$
are exact, then the functor t is said to be a right exact *functor.*

1.7 LEMMA. An additive functor $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$, covariant in the first and contravariant in the second variable, is right exact if and only if for every exact sequence $A_1' \to A_1 \to A_1'' \to 0$ in $\mathcal{A}_1$ and $0 \to A_2' \to A_2 \to A_2''$ in $\mathcal{A}_2$ the sequences
$$t(A_1', A_2) \longrightarrow t(A_1, A_2) \longrightarrow t(A_1'', A_2) \longrightarrow 0$$
and
$$t(A_1, A_2'') \longrightarrow t(A_1, A_2) \longrightarrow t(A_1, A_2') \longrightarrow 0$$
are exact.

Proof. "⇒": Let $A_1' \to A_1 \to A_1'' \to 0$ be an exact sequence in $\mathcal{A}_1$ and consider the commutative diagram



where $M' := \ker(A_1' \to A_1)$, $M := \ker(A_1 \to A_1'') = \operatorname{im}(A_1' \to A_1)$ (exactness of $A_1' \to A_1 \to A_1'' \to 0$) and with short exact sequences $0 \to M' \to A_1' \to M \to 0$ and $0 \to M \to A_1 \to A_1'' \to 0$.

Applying $t(\bullet, A_2)$ to this diagram yields the exact sequences $t(M', A_2) \to t(A_1', A_2) \to t(M, A_2) \to 0$ and $t(M', A_2) \to t(A_1, A_2) \to t(A_1'', A_2) \to 0$.

From the latter exact sequence it follows that the sequence

$$t(A_1', A_2) \to t(A_1, A_2) \to t(A_1'', A_2) \to 0 \tag{1.8}$$

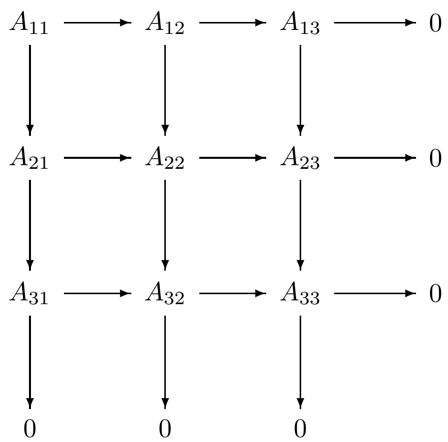
is exact at $t(A_1'', A_2)$.

From the commutativity of the triangle A_1', A_1, M and since the sequences $t(A_1', A_2) \to t(M, A_2) \to 0$ and $t(M, A_2) \to t(A_1, A_2) \to t(A_1'', A_2)$ are exact it follows that $\operatorname{im}(t(A_1', A_2) \to t(A_1, A_2)) = \operatorname{im}(t(M, A_2) \to t(A_1, A_2)) = \ker(t(A_1, A_2) \to t(A_1'', A_2))$. Therefore the sequence 1.8 is exact at $t(A_1, A_2)$, too, as had to be shown.

The proof for the second variable is similar.

"⇐": Clear. □

Another criterion for the right exactness of an additive functor needs a little bit of preparation: Let A_{kl} $(1 \leq k, l \leq 3)$ be objects of an abelian category $\mathcal{A}$ and assume that the diagram



(1.9)

is commutative and has exact columns and rows. From this diagram one can construct the sequence

$$A_{12} \oplus A_{21} \xrightarrow{\psi} A_{22} \xrightarrow{\varphi} A_{33} \longrightarrow 0 \tag{1.10}$$

where ψ is the sum of the homomorphisms $A_{12} \oplus A_{21} \to A_{12} \to A_{22}$ and $A_{12} \oplus A_{21} \to A_{21} \to A_{22}$, and φ the composition $A_{22} \to A_{23} \to A_{33}$ $(= A_{22} \to A_{32} \to A_{33})$.

1.11 Lemma. The sequence 1.10 is exact.

Proof. This lemma is proven by chasing members as described in appendix A.

Clearly φ is an epimorphism and $\varphi\psi = 0$. Hence it remains to be shown, that for every $x \in_m A_{22}$ such that $\varphi x \equiv 0$ there exists a $y \in_m A_{12} \oplus A_{21}$ such that $\psi y \equiv x$.

By the exactness of the lowest row there exists a $x_{31} \in_m A_{31}$ such that $(A_{31} \to A_{32}) \circ x_{31} \equiv (A_{22} \to A_{32}) \circ x$. Since $A_{21} \to A_{31}$ is an epimorphism there exists a $x_{21} \in_m A_{21}$ such that $(A_{21} \to A_{31}) \circ x_{21} \equiv x_{31}$. Define $x_{22} := (A_{21} \to A_{22}) \circ x_{21}$ and observe that due to the commutativity of the lower left square in the diagram $(A_{22} \to A_{32}) \circ x_{22} \equiv (A_{22} \to A_{32}) \circ x$. Hence $(A_{22} \to A_{32}) \circ (x - x_{22}) \equiv 0$. Now the middle column is exact and therefore there exists a $x_{12} \in_m A_{12}$ such that $(A_{12} \to A_{22}) \circ x_{12} \equiv x - x_{22}$.

Define $y := x_{12} + x_{21} \in_m A_{12} \oplus A_{21}$. Now $\psi \circ y = \psi \circ x_{12} + \psi \circ x_{21} = (A_{12} \to A_{22}) \circ x_{12} + (A_{21} \to A_{22}) \circ x_{21} \equiv x - x_{22} + x_{22} \equiv x$. □

With this preparation one can now prove the

1.12 Lemma. Let $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be an additive functor as in lemma 1.7. Then t is right exact if and only for every exact sequence $A_1' \to A_1 \to A_1'' \to 0$ in $\mathcal{A}_1$ and $0 \to A_2' \to A_2 \to A_2''$ in $\mathcal{A}_2$ the sequence

$$t(A_1', A_2) \oplus t(A_1, A_2'') \xrightarrow{\varphi} t(A_1, A_2) \longrightarrow t(A_1'', A_2') \longrightarrow 0 \tag{1.13}$$

is exact, where $\varphi := t(A_1' \to A_1, A_2 \to A_2'')$.

Proof. "⇒": Lemma 1.7 states that the commutative diagram

$$\begin{array}{ccccccc}
t(A_1', A_2'') & \longrightarrow & t(A_1, A_2'') & \longrightarrow & t(A_1'', A_2'') & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
t(A_1', A_2) & \longrightarrow & t(A_1, A_2) & \longrightarrow & t(A_1'', A_2) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
t(A_1', A_2') & \longrightarrow & t(A_1, A_2') & \longrightarrow & t(A_1'', A_2') & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

has exact rows and columns. Now the result follows follows from lemma 1.11.

"⇐": Given any objects A_k in $\mathcal{A}_k$, there exist exact sequences $0 \to A_1 \overset{\mathrm{id}}{\to} A_1 \to 0$ and $0 \to A_2 \overset{\mathrm{id}}{\to} A_2 \to 0$.

Hence we have by assumption the exact sequences

$$t(A_1', A_2) \longrightarrow t(A_1, A_2) \longrightarrow t(A_1'', A_2) \longrightarrow 0$$

and

$$t(A_1, A_2'') \longrightarrow t(A_1, A_2) \longrightarrow t(A_1, A_2') \longrightarrow 0$$

Hence by lemma 1.7 it follows that t is right exact. □

1.14 Remark. For left exactness exist similar lemmas such as lemma 1.7 and lemma 1.12. Just the direction of some arrows has to be adjusted. [CE56, pp. 23–27]

1.2 n-graded Complexes

Let $\mathcal{A}$ be an abelian category. The aim of this section is to define n-graded collections in $\mathcal{A}$ and to extend this definition to the concept of an n-graded complex in $\mathcal{A}$ which is a generalization of a chain complex (of modules).

In sequence objects will often be indexed by an n-tuple of integers. The "bar"-convention simplifies the notation:

Elements of the abelian group $\mathbb{Z}^n$ are denoted by variables with a bar on top, this is, $\bar{\imath} := (i_1, \ldots, i_n)$, $\bar{\jmath} := (j_1, \ldots, j_n)$, $\bar{k}$, $\bar{l}$, etc. $\bar{e}_k$ $(1 \leq k \leq n)$ denote the elements $\bar{e}_k := (0, \ldots, e_k = 1, \ldots 0)$, that is, $\bar{\imath} = i_1\bar{e}_1 + \ldots + i_n\bar{e}_n$ for all $\bar{\imath} \in \mathbb{Z}^n$. The total integer value of $\bar{\imath}$ is defined as $\sigma(\bar{\imath}) := i_1 + \ldots + i_n$. $\bar{\imath} \geq \bar{\jmath}$ is true if and only if $i_k \geq j_k$ for all $1 \leq k \leq n$, and similarly $\bar{\imath} > \bar{\jmath}$, $\bar{\imath} \leq \bar{\jmath}$ and $\bar{\imath} < \bar{\jmath}$ have to be understood.

1.15 Definition. *Let $\mathcal{A}$ be an abelian category. An* n-graded collection M in $\mathcal{A}$ *is a collection of objects $M_{\bar{\imath}}$ where the index runs through all n-tuples of integers, in symbols $M := (M_{\bar{\imath}})$. A* subcollection M' *of M is an n-graded collection $M' := (M'_{\bar{\imath}})$ such that $M'_{\bar{\imath}}$ is a subobject of $M_{\bar{\imath}}$ for each $\bar{\imath} \in \mathbb{Z}^n$, in symbols $M' \subset M$. If $M' \subset M$ as n-graded collections, then for every $\bar{\imath} \in \mathbb{Z}^n$ the quotient object $(M/M')_{\bar{\imath}} := M_{\bar{\imath}}/M'_{\bar{\imath}}$ is defined and we get a n-graded collection $M/M' := ((M/M')_{\bar{\imath}})$ called the* quotient collection of M and M'.

A n-graded collection M is said to be positive *(respectively* negative*) if $M_{\bar{\imath}} = 0$ whenever $\bar{\imath} \ngeq 0$ (respectively $\bar{\imath} \nleq 0$).*

If M, M' are two n-graded collections in $\mathcal{A}$, then a homomorphism $f : M \to M'$ of n-graded collections in $\mathcal{A}$ of degree $\bar{p}$ *is a collection of homomorphisms $f := \{f_{\bar{\imath}} : M_{\bar{\imath}} \to M'_{\bar{\imath}+\bar{p}}\}$ where the index runs through all n-tuples of integers. Composition is performed component-wise. The set of all homomorphisms $f : M \to M'$ of n-graded collections of degree $\bar{p}$ is denoted by* $\mathrm{Hom}_{\bar{p}}(M, M')$*, the set of all homomorphisms $f : M \to M'$ of any degree $\bar{p}$ is denoted by* $\mathrm{Hom}(M, M')$.

Given a homomorphism $f : M \to M'$ of n-graded collections of degree $\bar{p}$ there are two subcollections related to f, the *image of f* and the *kernel of f*

$$\mathrm{im}(f) := (\mathrm{im}(f_{\bar{\imath}-\bar{p}}))$$
$$\ker(f) := (\ker(f_{\bar{\imath}}))$$

and two quotient-collections, the *coimage of f* and the *cokernel of f*

$$\begin{aligned} \operatorname{coim}(f) &:= M/\ker(f) \\ \operatorname{coker}(f) &:= M'/\operatorname{im}(f) \end{aligned}$$

Actually one has to be a little bit more careful with the meaning of *the* image, *the* kernel etc. This is since one can define these objects in general abelian categories only up to isomorphism. For example, to be exact *a* (and not *the*) kernel of a homomorphism $f\colon M \to M'$ is a monomorphism $k\colon K \to M$ of degree zero such that $f \circ k = 0$ and every homomorphism $g\colon M'' \to M$ such that $f \circ g = 0$ factors in an unique way through k. Similar more careful definitions for the image, cokernel and coimage can be constructed (see for example [Mit65]).

Nonetheless I will use in this thesis the more sloppy notation for these objects, as long as there is no danger of misinterpretation. For example, in the case of

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

$\operatorname{im}(f) = \ker(g)$ means actually, that any kernel $k\colon K \to M$ of g is actually also an image of f. The homomorphisms are then kept fixed while drawing conclusions from this statement.

As a side-note, n-graded collections in $\mathcal{A}$ and homomorphisms of degree 0 form a category denoted by $\mathbf{G}^n\mathcal{A}$ or just $\mathbf{G}\mathcal{A}$ in the single-graded case (that is $n = 1$).

Consider the case $\mathcal{A} = R$–**Mod**. Recall, that a single-graded collection $C = (C_{\bar{\imath}})$ is endowed with the structure of a chain complex by introducing a differentiation homomorphism $\partial : C \to C$ of degree -1. Homomorphisms of chain complexes (called chain maps) are required to commute with the differentiation homomorphism.

In the more general case of an n-graded collection in an arbitrary abelian category $\mathcal{A}$ one differentiation homomorphism is not enough ($n > 1$), but n such homomorphism are needed. This leads to the

1.16 Definition. *Let $C = (C_{\bar{\imath}})$ be an n-graded collection in $\mathcal{A}$. A* set of differentiations *$\partial := \{\partial^{(k)}\}$* on *$C$ is a collection of n homomorphisms $\partial^{(k)} : C \to C$ of degree $-\bar{e}_k$ ($1 \le k \le n$) which anti-commute, that is for every $1 \le k, k' \le n$*

$$\partial^{(k)}\partial^{(k')} + \partial^{(k')}\partial^{(k)} = 0 \quad \textit{and} \qquad \partial^{(k)}\partial^{(k)} = 0$$

An n-graded collection in $\mathcal{A}$ together with a set of differentiations ∂ is a n-graded complex in $\mathcal{A}$*, in symbols $C := (C_{\bar{\imath}}, \partial^{(k)})$ or just $C := (C_{\bar{\imath}}, \partial)$. A single-graded complex is called a* chain complex*.*

Assume that $C' = (C'_{\bar{\imath}})$ is a subcollection of an n-graded complex $C = (C_{\bar{\imath}}, \partial)$, and denote the inclusion $C' \to C$ by i. If for every $1 \le k \le n$ there exists a $\partial'^{(k)}$ such that $\partial^{(k)} i = i\partial'^{(k)}$, then $C' = (C'_{\bar{\imath}}, \partial')$ (or $C' = (C'_{\bar{\imath}}, \partial)$ by abuse of notation) is a subcomplex *of C, in symbols $C' \subset C$. If $C' \subset C$, then the set of differentiations ∂ on C induce a set of differentiations $\bar{\partial}$ on the quotient collection C/C'. The complex $C/C' := ((C/C')_{\bar{\imath}}, \bar{\partial})$ is the* quotient complex of C and C'*.*

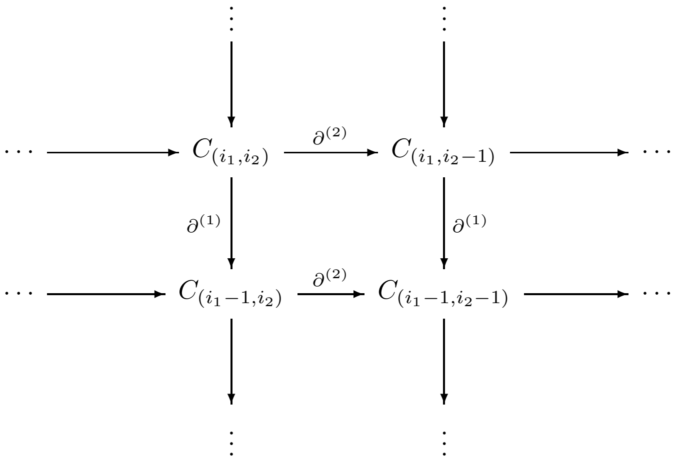


Figure 1: A fragment of a bi-graded complex in an abelian category $\mathcal{A}$.

If $C = (C_{\bar{\imath}}, \partial)$ and $C' = (C'_{\bar{\imath}}, \partial')$ are two n-graded complexes, then a homomorphism $f\colon C \to C'$ of n-graded collections of degree $\bar{p}$ is a homomorphism of n-graded complexes of degree $\bar{p}$ *if f commutes with the differential operators in the way that*

$$f\partial^{(k)} = (-1)^{\sigma(\bar{p})}\,\partial'^{(k)} f$$

for all $1 \leq k \leq n$. The set of all homomorphisms $f\colon C \to C'$ of n-graded complexes of degree $\bar{p}$ is denoted by $\operatorname{Hom}_{\bar{p}}(C, C')$, and the set of all homomorphisms $f\colon C \to C'$ of any degree $\bar{p}$ is denoted by $\operatorname{Hom}(C, C')$. A homomorphism of single-graded complexes is called a chain map.

If $f\colon C \to C'$ is a homomorphism of n-graded complexes in $\mathcal{A}$, then $\operatorname{im}(f)$, $\ker(f)$, $\operatorname{coim}(f)$ and $\operatorname{coker}(f)$ are n-graded complexes.

Again, n-graded complexes in $\mathcal{A}$ and homomorphisms of degree 0 form a category, which is denoted by $\partial^n\mathcal{A}$ or just $\partial\mathcal{A}$ in the single-graded case.

In the single-graded case the differentiation ∂ gives rise to the definition of the *cycles* and *cocycles* ($Z(C) := \ker(\partial)$ and $Z' := \operatorname{coker}(\partial)$), *boundaries* and *coboundaries* ($B(C) := \operatorname{im}(\partial)$ and $B'(C) := \operatorname{coim}(\partial)$) and, since $\operatorname{im}(\partial) \subset \ker(\partial)$, *homology* $H(C) := \ker(\partial)/\operatorname{im}(\partial)$ of the complex C (see figure 2). By construction these objects are chain complexes with trivial differentiation and these relations are functorial.

In the n-graded case ($n > 1$) these objects cannot be defined directly. But one can try to construct for an n-graded complex C an associated single-graded complex $\tilde{C}$ and translate this way the previously defined objects to n-graded complexes.

1.17 Definition. *Let $\mathcal{A}$ be an abelian category and $C = (C_{\bar{\imath}}, \partial)$ an n-graded*

$$
\begin{aligned}
Z(C) &:= \ker(\partial) \\
Z'(C) &:= \operatorname{coker}(\partial) = C/\operatorname{im}(\partial) \\
B(C) &:= \operatorname{im}(\partial) \\
B'(C) &:= \operatorname{coim}(\partial) = C/\ker(\partial) \\
H(C) &:= \ker(\partial)/\operatorname{im}(\partial)
\end{aligned}
$$

Figure 2: The definition of cycles, boundaries etc.

complex. Assume that the the objects

$$\tilde{C}_i := \sum_{\sigma(\bar{\imath})=i} C_{\bar{\imath}} \tag{1.18}$$

exists for all $i \in \mathbb{Z}$. Then the $\tilde{C}_i$ define a single-graded collection in $\mathcal{A}$, which becomes a single-graded complex $\tilde{C} = (\tilde{C}_i, \tilde{\partial})$ with $\tilde{\partial}$ being the differentiation induced by

$$\tilde{\partial}_{\bar{\imath}} := \partial^{(1)}_{\bar{\imath}} + \ldots + \partial^{(n)}_{\bar{\imath}}$$

The complex $\tilde{C}$ is called the associated (single-graded) complex *of the n-graded complex C.*

If for a given n-graded complex C the right side of 1.18 is defined for all $i \in \mathbb{Z}$, then the n-graded complex C is said to have *or* admit an associated complex.

Since an abelian categories $\mathcal{A}$ has finite coproducts, any positive or negative n-graded complex admits an associated complex. If the abelian category $\mathcal{A}$ has coproducts in general (like the category of left R-modules R–**Mod** or the category of abelian groups **AG**) then any n-graded complex admits an associated complex. A single-graded complex C admits always a associated complex $\tilde{C}$ and clearly $C = \tilde{C}$.

If C is an n-graded complex in an abelian category with coproducts then the functors Z, Z', B, B' and H are understood to be applied to the associated complex $\tilde{C}$ of C, that is $Z(C) := Z(\tilde{C})$, $Z'(C) := Z'(\tilde{C})$, $B(C) := B(\tilde{C})$, $B'(C) := B'(\tilde{C})$ and $H(C) := H(\tilde{C})$.

An n-graded complex C is said to be *acyclic* if it admits an associated complex and $H(C) = 0$.

Consider the single-graded case. Two chain maps $f, g\colon C \to C'$ of degree 0 are said to be *homotopic* if there exists a homomorphism of collections $D\colon C \to C'$ of degree 1 (called *homotopy*) such that $D\partial + \partial' D = f - g$, in symbols $D\colon f \simeq g$ or just $f \simeq g$. Homotopic chain maps induce the same homomorphism $f_* = g_*\colon H(C) \to H(C')$.

In the general case if $n > 1$ the homotopy homomorphism has to be replaced by a set of n homotopy homomorphisms. The definition is analogous to the general definition of the differentiations:

1.19 DEFINITION. *Let $f, g\colon C \to C'$ be homomorphisms of degree 0 of n-graded complexes. f is said to be homotopic to g (in symbols $f \simeq g$) if there exists a set of n homomorphisms of n-graded collections $D^{(k)}\colon C \to C'$ (called* homotopies*) of degree $\bar{e}_k$ such that*

$$\sum_{k=1}^{n} D^{(k)}\partial^{(k)} + \partial'^{(k)}D^{(k)} = f - g$$

and

$$D^{(k)}\partial^{(k')} + \partial'^{(k')}D^{(k)} = 0 \qquad (k \neq k')$$

Homotopic homomorphisms $f, g\colon C \to C'$ induce homotopic homomorphisms on the associated complexes $\tilde{f}, \tilde{g}\colon \tilde{C} \to \tilde{C}'$ and hence the same homomorphisms $f_* = g_*\colon H(C) \to H(C')$.

Two complexes C and C' are said to be *homotopic* if there exists homomorphisms $f\colon C \to C'$ and $g\colon C' \to C$ such that $fg \simeq \mathrm{id}_{C'}$ and $gf \simeq \mathrm{id}_C$.

1.3 Functors of Complexes

Consider as an example the tensor product, which is an additive functor

$$\otimes\colon \textbf{Mod}\text{–}R \times R\text{–}\textbf{Mod} \to \textbf{AG}$$

covariant in both variables. Let $C^{(1)} = (C_i^{(1)}, \partial^{(1)})$ be a chain complex of right R-modules and $C^{(2)} = (C_i^{(2)}, \partial^{(2)})$ be a chain complex of left R-modules. Then

$$T_{\bar{\imath}} := C_{i_1}^{(1)} \otimes C_{i_2}^{(2)}$$

defines a bi-graded collection T of abelian groups. The aim is to extend the bi-graded collection to a bi-graded complex using the differentiations of the complexes $C^{(1)}$ and $C^{(2)}$.

The naive approach would be to define the set of differentiations ∂ on T as

$$\partial_{\bar{\imath}}^{(1)} := \partial_{i_1}^{(1)} \otimes \mathrm{id} \qquad \text{and} \qquad \partial_{\bar{\imath}}^{(2)} := \mathrm{id} \otimes \partial_{i_2}^{(2)}$$

One obtains two homomorphism of degree $-\bar{e}_1$ and $-\bar{e}_2$. But although $\partial^{(1)}\partial^{(1)}$ and $\partial^{(2)}\partial^{(2)}$ are both trivial, this approach fails, because the differentiations do not anti-commute but commute:

$$\begin{aligned}\partial_{\bar{\imath}-\bar{e}_2}^{(1)}\partial_{\bar{\imath}}^{(2)} + \partial_{\bar{\imath}-\bar{e}_1}^{(2)}\partial_{\bar{\imath}}^{(1)} &= (\partial_{i_1}^{(1)} \otimes \mathrm{id})(\mathrm{id} \otimes \partial_{i_2}^{(2)}) + (\mathrm{id} \otimes \partial_{i_2}^{(2)})(\partial_{i_1}^{(1)} \otimes \mathrm{id}) \\ &= \partial_{i_1}^{(1)} \otimes \partial_{i_2}^{(2)} + \partial_{i_1}^{(1)} \otimes \partial_{i_2}^{(2)}\end{aligned}$$

One has to introduce an alternating sign to achieve anti-commutativity:

$$\partial_{\bar{\imath}}^{(1)} := \partial_{i_1}^{(1)} \otimes \mathrm{id} \qquad \text{and} \qquad \partial_{\bar{\imath}}^{(2)} := (-1)^{i_1}\, \mathrm{id} \otimes \partial_{i_2}^{(2)}$$

With this new definitions two homomorphism of degree $-\bar{e}_1$ and $-\bar{e}_2$ respectively are obtained. Still one has $\partial^{(1)}\partial^{(1)} = 0$ and $\partial^{(2)}\partial^{(2)} = 0$, but now those homomorphisms anti-commute as desired:

$$\begin{aligned}\partial^{(1)}_{\bar{\imath}-\bar{e}_2}\partial^{(2)}_{\bar{\imath}} + \partial^{(2)}_{\bar{\imath}-\bar{e}_1}\partial^{(1)}_{\bar{\imath}} &= (-1)^{i_1}(\partial^{(1)}_{i_1} \otimes \mathrm{id})(\mathrm{id} \otimes \partial^{(2)}_{i_2}) \\ &\qquad + (-1)^{i_1-1}(\mathrm{id} \otimes \partial^{(2)}_{i_2})(\partial^{(1)}_{i_1} \otimes \mathrm{id}) \\ &= (-1)^{i_1}\big(\partial^{(1)}_{i_1} \otimes \partial^{(2)}_{i_2} - \partial^{(1)}_{i_1} \otimes \partial^{(2)}_{i_2}\big) \\ &= 0\end{aligned}$$

This example motivates the following general definition

1.20 Definition. *Let $t\colon \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \to \mathcal{A}$ be an additive functor of abelian categories, covariant in some and contravariant in the other variables. Let $A^{(k)}$ be a single-graded collections in $\mathcal{A}_k$ $(1 \le k \le n)$. Then $t(A^{(1)}, \ldots, A^{(n)})$ is defined to be the n-graded collection defined by*

$$t(A^{(1)}_{\varepsilon_1 i_1}, \ldots, A^{(n)}_{\varepsilon_n i_n})$$

where $\varepsilon_k = +1$ if t is covariant in the k-th variable and $\varepsilon_k = -1$ if t is contravariant in the k-th variable.

If A'_k $(1 \le k \le n)$ are single-graded collections in $\mathcal{A}_k$, and $f^{(k)} : A_k \to A'_k$ (if the k-th variable of t is covariant) or $f^{(k)} : A'_k \to A_k$ (if the k-th variable of t is contravariant) are homomorphisms of degree p_k, then $t(f^{(1)}, \ldots, f^{(n)})$ is the homomorphism of degree $\bar{p}$ of n-graded collections in $\mathcal{A}$

$$t(f^{(1)}, \ldots, f^{(n)})\colon t(A^{(1)}, \ldots, A^{(n)}) \to t(A'^{(1)}, \ldots, A'^{(n)})$$

defined component wise as

$$t_{\bar{\imath}}(f^{(1)}, \ldots, f^{(n)}) := (-1)^{\varepsilon} t(f^{(1)}_{l_1}, \ldots, f^{(n)}_{l_n})$$

where $\varepsilon := \sum_{j<k} i_j p_k$, $l_k = i_k$ if the k-th variable of t is covariant and $l_k = -(i_k + p_k)$ if the k-th variable of t is contravariant.

If $g^{(k)}\colon A'^{(k)} \to A''^{(k)}$ (respectively $g^{(k)}\colon A''^{(k)} \to A'^{(k)}$) are homomorphisms of degree q_k and $h^{(k)} = g^{(k)} f^{(k)}$ (respectively $h^{(k)} = f^{(k)} g^{(k)}$), then one can show [CE56, p. 63] that

$$t(h^{(1)}, \ldots, h^{(n)}) = (-1)^{\eta} t(g^{(1)}, \ldots, g^{(n)})\, t(f^{(1)}, \ldots, f^{(n)})$$

where $\eta = \sum_{j<k} p_j q_k$.

If all the $f^{(k)}$ and $g^{(k)}$ have degree 0 then the $h^{(k)}$ have degree 0, too, and $\eta = 0$. Therefore the mapping

$$\begin{aligned}(A^{(1)}, \ldots, A^{(n)}) &\mapsto t(A^{(1)}, \ldots, A^{(n)}) \\ (f^{(1)}, \ldots, f^{(n)}) &\mapsto t(f^{(1)}, \ldots, f^{(n)})\end{aligned}$$

is functorial for chain maps of degree 0. In this sense the the additive functor $t\colon \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \to \mathcal{A}$ is extended to a functor

$$t\colon \mathbf{G}\mathcal{A}_1 \times \ldots \times \mathbf{G}\mathcal{A}_n \to \mathbf{G}^n\mathcal{A}$$

If the abelian category $\mathcal{A}$ has coproducts, one can pass to the associated single-graded collection and gets a functor

$$t\colon \mathbf{G}\mathcal{A}_1 \times \ldots \times \mathbf{G}\mathcal{A}_n \to \mathbf{G}\mathcal{A}$$

If now $C^{(k)} := (C^{(k)}_{\bar{\imath}}, \partial^{(k)})$ are single-graded complexes in $\mathcal{A}_k$, then (by abuse of notation)

$$\partial^{(k)} := t(\mathrm{id}, \ldots, \partial^{(k)}, \ldots, \mathrm{id}) \qquad (1 \leq k \leq n) \tag{1.21}$$

anti-commute and define a set of differentiations on the n-graded collection $t(C^{(1)}, \ldots, C^{(n)})$.

1.22 Definition. *If $t\colon \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \to \mathcal{A}$ is an additive functor of abelian categories and $C^{(k)} := (C^{(k)}_{\bar{\imath}}, \partial^{(k)})$ are single-graded complexes in $\mathcal{A}_k$ $(1 \leq k \leq n)$, then $t(C^{(1)}, \ldots, C^{(n)})$ denotes the n-graded complex*

$$t(C^{(1)}, \ldots, C^{(n)}) := (t_{\bar{\imath}}(C^{(1)}, \ldots, C^{(n)}), \partial^{(k)})$$

where the differentiations $\partial^{(k)}$ are defined as in equation 1.21.

In this way the additive functor $t\colon \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \to \mathcal{A}$ is extended to a functor

$$t\colon \partial\mathcal{A}_1 \times \ldots \times \partial\mathcal{A}_n \to \partial^n\mathcal{A}$$

Again, if the abelian category $\mathcal{A}$ has coproducts, one can pass to the associated single-graded complex and gets a functor

$$t\colon \partial\mathcal{A}_1 \times \ldots \times \partial\mathcal{A}_n \to \partial\mathcal{A}$$

If $D^{(k)}\colon f^{(k)} \simeq f'^{(k)}$ are homotopies $(1 \leq k \leq n)$, then by abuse of notation $D^{(k)} := t(\mathrm{id}, \ldots, D^{(k)}, \ldots, \mathrm{id})$ one obtains a homotopy

$$D\colon t(f^{(1)}, \ldots, f^{(n)}) \simeq t(f'^{(1)}, \ldots, f'^{(n)})$$

In particular homotopic chain complexes are mapped to homotopic complexes and homotopic chain maps to homotopic homomorphisms of n-graded complexes.

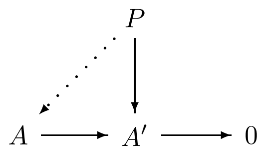
2 Derived Functors

Derived functors appear while studying the effect of additive functors on the homology of the most basic chain complex: projective and injective resolutions. Resolutions of short exact sequences together with the Snake Lemma then yield long exact sequences involving the left derived functors t_i (or right derived functors t'_i).

From this result only a tiny piece is needed later when proving the Künneth Formula (see theorem 3.19 and 3.21), namely the exactness of the sequence $t_1(X, Y) \to t_1(X'', Y) \to t(X', Y)$ in the case of t being a right exact functor and $t(X', Y) \to t'_1(X'', Y) \to t'_1(X, Y)$ in the case of t being a left exact functor.

2.1 Resolutions

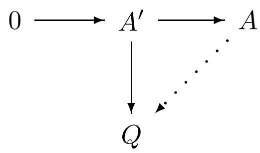
2.1 Definition. *An object P in a category $\mathcal{A}$ is* projective *if for every diagram*



with exact row there is a morphism $P \to A$ making the diagram commutative.

$\mathcal{A}$ is said to have projectives *if each object A in $\mathcal{A}$ is a quotient object of a projective object P in $\mathcal{A}$.*

An object Q in $\mathcal{A}$ is injective *if for every diagram*



with exact row there is a morphism $A \to Q$ making the diagram commutative.

If every object in $\mathcal{A}$ is a subobject of a injective object Q, the category is said to have injectives.

There are certain chain complexes which will be drawn to our attention in the following. These are chain complexes with a very simple homology (with only one non-trivial homology object at dimension zero) and each object being projective (or injective):

2.2 Definition. *Let $\mathcal{A}$ be an abelian category and A an object of $\mathcal{A}$.*

An augmentation over *A of a positive chain complex P in $\mathcal{A}$ is a homomorphism $\varepsilon : P_0 \to A$ such that $\varepsilon\partial_1 = 0$. A* (positive) resolution of *A is a positive chain complex P augmented over A such that the sequence*

$$\ldots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

is exact. If every P_k *is projective then* P *is a* projective resolution*. If every object of* $\mathcal{A}$ *admits a projective resolution then the category* $\mathcal{A}$ *is said to* have projective resolutions*.*

Similarly, an augmentation over A *of a negative chain complex* Q *in* $\mathcal{A}$ *is a homomorphism* $\varepsilon : A \to Q_0$ *such that* $\partial_0 \varepsilon = 0$*. A* (negative) resolution of A *is a negative chain complex* Q *augmented over* A *such that the sequence*

$$0 \longrightarrow A \xrightarrow{\varepsilon} Q_0 \longrightarrow Q_{-1} \longrightarrow Q_{-2} \longrightarrow Q_{-3} \longrightarrow \ldots$$

is exact. If every Q_k *is injective then* Q *is a* injective resolution*. If every object of* $\mathcal{A}$ *admits a injective resolution then the category* $\mathcal{A}$ *is said to* have injective resolutions*.*

If P is a positive resolution of A, the augmentation homomorphism is a epimorphism. Dually, if Q is a negative resolution then the augmentation homomorphism is a monomorphism.

2.3 Lemma. An abelian category $\mathcal{A}$ has projective resolutions if and only if it has projectives. Similarly, $\mathcal{A}$ has injective resolutions if and only if it has injectives.

Proof. The proof is stated only for the projective case, the injective case is similar:

"$\Rightarrow$": Let A be an object of $\mathcal{A}$ and P a projective resolution of A. Then P_0 is projective, $P_0 \to A$ is a epimorphism and hence A is a quotient object of P_0. Therefore $\mathcal{A}$ has projectives.

"$\Leftarrow$": Let A be an object in $\mathcal{A}$. Then A is the quotient object of some projective object P_0, that is there exists an exact sequence $P_0 \to A \to 0$. Let K_0 be the kernel of $P_0 \to A$. Then K_0 is the quotient object of some projective object P_1. Let $P_1 \to P_0$ be the composition $P_1 \to K_0 \to P_0$. Then the sequence $P_1 \to P_0 \to A$ is exact by construction. Iterating this construction results in a projective resolution of A. □

2.4 Definition. *Let* $\mathcal{A}$ *be an abelian category,* A *and* A' *two objects of* $\mathcal{A}$*, and* $f\colon A \to A'$ *a homomorphism. Furthermore, let* X *(respectively* X'*) be a positive chain complex in* $\mathcal{A}$ *augmented over* A *(respectively* A'*). A chain map* $F\colon X \to X'$ *that makes the diagram*

$$\begin{array}{ccc} X & \xrightarrow{F} & X' \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon'} \\ A & \xrightarrow[f]{} & A' \end{array}$$

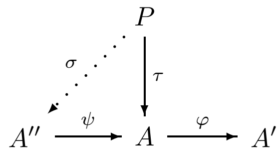
commutative is said to be a homomorphism over f*.*

2.5 Proposition. Let X be a projective positive chain complex augmented over A, X' a positive resolution of A' and $f\colon A \to A'$ a homomorphism. Then there exists a homomorphism $F\colon X \to X'$ over f and any two such homomorphisms are homotopic.

In particular this applies if X and X' are projective resolutions.

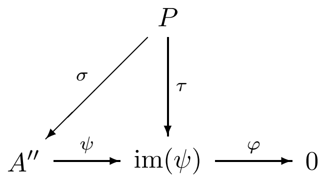
To prove this proposition the following lemma is needed:

2.6 Lemma. Consider the diagram



with P projective and the row exact. If $\varphi\tau = 0$, then there exists a homomorphism $\sigma\colon P \to A''$ such that $\tau = \psi\sigma$.

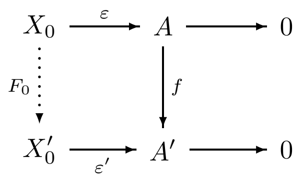
Proof. Since $\varphi\tau = 0$ there exists the inclusion $\mathrm{im}(\tau) \subset \ker(\varphi)$. Due to the exactness of the row $A'' \to A \to A'$, $\ker(\varphi) = \mathrm{im}(\psi)$, that is $\mathrm{im}(\tau) \subset \mathrm{im}(\psi)$. As P is projective, there exists a σ making the diagram



commutative. Clearly σ is the desired homomorphism. □

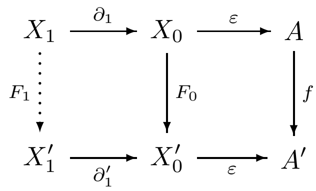
Proof of proposition 2.5. The homomorphism F is constructed recursively as follows:

Consider the diagram



Since X_0 is projective and the lower row is exact, there exists a homomorphism $F_0\colon X_0 \to X_0'$ such that $\varepsilon' F_0 = f\varepsilon$.

Hence one gets the commutative diagram

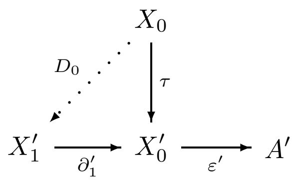


with the second row exact. Since $\varepsilon F_0 \partial_1 = f\varepsilon\partial_1 = 0$ and X_1 is projective, one can apply lemma 2.6 and gets a homomorphism $F_1\colon X_1 \to X_1'$ such that $\partial_1' F_1 = F_0 \partial_1$.

Iterating this construction one gains recursively homomorphisms $F_i\colon X_i \to X_i'$ $(i > 0)$ and by this a homomorphism $F\colon X \to X'$ over f.

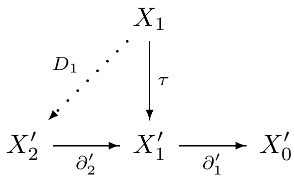
This proves the existence of a homomorphism F over f. In order to prove the uniqueness up to homotopy one needs to construct a homotopy $D\colon F \simeq F'$ for two given homomorphisms $F, F'\colon X \to X'$ over f.

Therefore one considers first the diagram



where $\tau := F_0' - F_0$. Now $\varepsilon'\tau = \varepsilon'(F_0' - F_0) = \varepsilon' F_0' - \varepsilon' F_0 = f\varepsilon - f\varepsilon = 0$ and hence (lemma 2.6) there is a homomorphism $D_0\colon X_0 \to X_1'$ such that $\partial_1' D_0 = \tau$. The homomorphism ∂_0 is by assumption trivial and therefore one gets $\partial_1' D_0 + D_{-1}\partial_0 = F_0' - F_0$.

Next consider the diagram



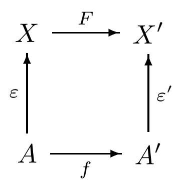
with $\tau := F_1' - F_1 - D_0\partial_1$. As similarly above, $\partial_1'\tau = 0$, and hence by lemma 2.6 there exists a homomorphism $D_1\colon X_1 \to X_2'$ such that $\partial_2' D_1 = \tau$, that is $\partial_2' D_1 + D_0\partial_1 = F_1' - F_1$.

Iterating this construction yields the desired homotopy $D\colon F \simeq F'$. □

Now, if X and X' are two projective resolutions of the same object A of the abelian category $\mathcal{A}$, then it follows from proposition 2.5 that X and X' are homotopic. That is there exist homomorphisms $F\colon X \to X'$ and $F'\colon X' \to X$ over the identity homomorphism id_A, such that $FF' \simeq \mathrm{id}_X$ and $F'F \simeq \mathrm{id}_{X'}$.

Dually one makes for augmented negative chain complexes the

2.7 Definition. *Let X be a negative chain complex augmented over A (with augmentation ε) and X' be a negative chain complex augmented over A' (with augmentation ε'). A chain map $F\colon X \to X'$ is a* homomorphism over the homomorphism $f\colon A \to A'$, *if the diagram*



is commutative.

Similarly as above one proves the dual to proposition 2.5:

2.8 PROPOSITION. Let X be a negative resolution of A, X' a negative chain complex augmented over A' and $f\colon A \to A'$ a homomorphism. Then there exists a homomorphism $F\colon X \to X'$ over f and any two such maps are homotopic.

In particular this applies if X and X' are injective resolutions. □

Again, two injective resolutions X and X' of the same object A are homotopic.

2.2 Resolutions of Short Exact Sequences

Consider an abelian category $\mathcal{A}$ with projectives. If $0 \to A' \xrightarrow{\psi} A \xrightarrow{\varphi} A \to 0$ is a short exact sequence in $\mathcal{A}$, then the previous section stated that there exist projective resolutions X', X and X'' over A', A and A'' respectively.

On the other hand, there might exist homomorphisms Ψ and Φ over ψ and φ respectively, which yield a short exact sequence $0 \to X' \to X \to X'' \to 0$. This motivates the following

2.9 DEFINITION. *Assume that X', X and X'' are positive chain complexes augmented over A', A and A''. Assume further, that the sequence*

$$0 \longrightarrow A' \xrightarrow{\psi} A \xrightarrow{\varphi} A'' \longrightarrow 0 \tag{2.10}$$

is a short exact sequence. If there exist homomorphisms $\Psi\colon X' \to X$ and $\Phi\colon X \to X''$ over ψ and φ respectively, such that

$$0 \longrightarrow X' \xrightarrow{\Psi} X \xrightarrow{\Phi} X'' \longrightarrow 0 \tag{2.11}$$

is a short exact sequence, then the sequence 2.11 is called a (positive) sequence over the sequence 2.10.

If X', X and X'' are projective resolutions of A', A and A'' respectively, then the sequence 2.11 is said to be a (projective) resolution of the sequence 2.10.

Similarly *a (negative) sequence over* and *injective resolution of the sequence 2.10* are defined.

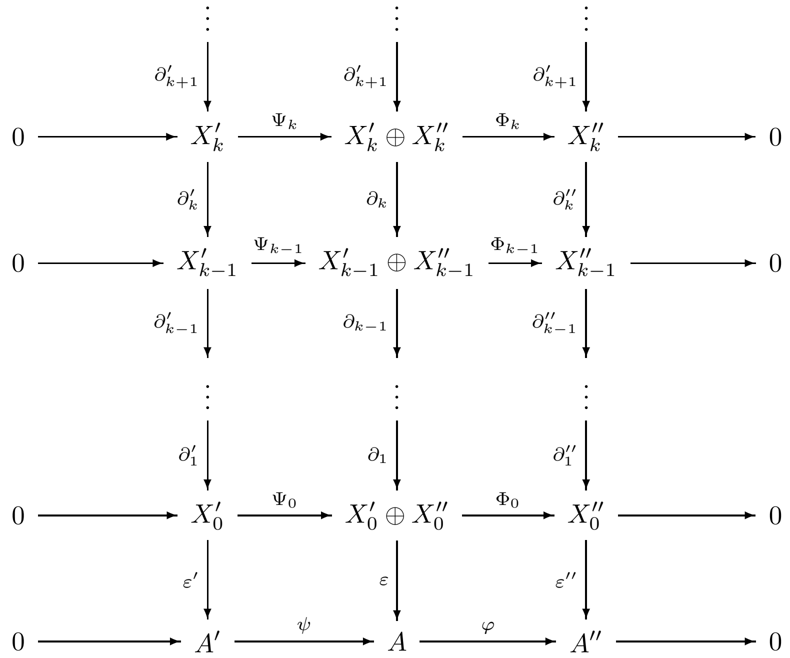
I will next show that given a short exact sequence 2.10 there exists projective (injective) resolutions if the category $\mathcal{A}$ has projectives (injectives respectively). Only the results for the projective case are state and proven, as again the injective case is analogous.

Some preparational work is needed:

2.12 DEFINITION. *A short exact sequence 2.11 is said to be* normal *if for each $k \in \mathbb{Z}$ the sequence $0 \to X'_k \to X_k \to X''_k \to 0$ is split.*

If the sequence 2.11 is normal, X can be replaced by the direct sum $X' \oplus X''$ and there exists homomorphisms $\Psi'\colon X' \oplus X'' \to X'$ and $\Phi'\colon X' \oplus X'' \to X''$ such that $\Psi'\Psi = \mathrm{id}_{X'}$, $\Phi\Phi' = \mathrm{id}_{X''}$, $\Psi\Phi = 0$, $\Phi'\Psi' = 0$ and $\Psi\Psi' + \Phi'\Phi = \mathrm{id}_{X' \oplus X''}$.

Assume that the sequence 2.11 is normal and is a resolution of the sequence 2.10. Then one has the situation



(2.13)

The aim is to describe the homomorphisms $\partial_k\colon X'_k \oplus X''_k \rightarrow X'_{k-1} \oplus X''_{k-1}$ and $\varepsilon\colon X'_0 \oplus X''_0 \rightarrow A$. The homomorphisms are uniquely defined by their coordinate homomorphisms and constraints can be derived through the commutativity of the diagram 2.13.

Consider first the augmentation homomorphism ε. It can be written as $\varepsilon = \psi\varepsilon'\Psi'_0 + \sigma\Phi_0$ with a homomorphism $\sigma\colon X''_0 \rightarrow A$. Now $\varepsilon''\Phi_0 = \varphi\varepsilon = \varphi(\psi\varepsilon'\Psi'_0 + \sigma\Phi_0) = \varphi\psi\varepsilon'\Psi'_0 + \varphi\sigma\Phi_0$ and hence (since $\varphi\psi = 0$) one has $\varepsilon''\Phi_0 = \varphi\sigma\Phi_0$, that is $\varepsilon'' = \varphi\sigma$ (since Φ_0 is an epimorphism).

Next one considers the homomorphism ∂_1. It can be written as

$$\partial_1 = \Psi_0\partial'_1\Psi'_1 + \Phi'_0\Theta'_1\Psi'_1 + \Psi_0\Theta_1\Phi_1 + \Phi'_0\partial''_1\Phi_1 \tag{2.14}$$

where $\Theta_1\colon X''_1 \rightarrow X'_0$ and $\Theta'_1\colon X'_1 \rightarrow X''_0$. Now one requirement for ∂_1 is that $\Phi_0\partial_1 = \partial''_1\Phi_1$, that is

$$\partial''_1\Phi_1 = \underbrace{\Phi_0\Psi_0}_{=0}\partial'_1\Phi'_1 + \underbrace{\Phi_0\Phi'_0}_{=\mathrm{id}}\Theta'_1\Psi'_1 + \underbrace{\Phi_0\Psi_0}_{=0}\Theta_1\Phi_1 + \underbrace{\Phi\Phi'_0}_{=\mathrm{id}}\partial''_1\Phi_1$$

Hence $\Theta'_1\Phi_1$ has to be trivial, which implies that $\Theta'_1 = 0$. The other constraint for ∂_1 is the relation $\varepsilon\partial_1 = 0$, which evaluates to

$$0 = (\varphi\varepsilon'\Psi'_0 + \sigma\Phi_0)(\Psi_0\partial'_1\Psi'_1 + 0 + \Psi_0\Theta_1\Phi_1 + \Phi'_0\partial''_1\Phi_1) = (\varphi\varepsilon'\Theta_1 + \sigma\partial''_1)\Phi_1$$

and therefore (since Φ_1 is a epimorphism) it must be that $\varphi\varepsilon'\Theta_1 + \sigma\partial''_1 = 0$.

Similarly one deduces from the constraint $\partial_{k-1}\partial_k = 0$ ($k > 1$) that ∂_k is of the form

$$\partial_k = \Psi_{k-1}(\partial'_k \Psi'_k + \Theta_k \Phi_k) + \Phi'_{k-1}\partial''_k \Phi_k$$

where the homomorphism $\Theta_k\colon X''_k \to X'_{k-1}$ satisfies the condition $\partial'_{k-1}\Theta_k + \Theta_{k-1}\partial''_k = 0$.

This motivates the

2.15 Definition. *Let*

$$0 \longrightarrow X' \xrightarrow{\Psi} X \xrightarrow{\Phi} X'' \longrightarrow 0 \tag{2.16}$$

be a normal sequence over the short exact sequence $0 \to A' \xrightarrow{\psi} A \xrightarrow{\varphi} A \to 0$*.*

The description

$$\begin{aligned} \partial_k &= \Psi_{k-1}(\partial'_k \Psi'_k + \Theta_k \Phi) + \Phi'_{k-1}\partial''_k \Phi_k \\ \varepsilon &= \psi \varepsilon' \Psi'_0 + \sigma \Phi_0 \end{aligned}$$

with $\sigma\colon X''_0 \to A$ *and* $\Theta_k\colon X''_k \to X'_{k-1}$ ($k > 0$) *being homomorphisms satisfying the conditions*

$$\left.\begin{aligned} \varepsilon'' &= \varphi\sigma \\ \varphi\varepsilon'\Theta_1 + \sigma\partial''_1 &= 0 \\ \partial'_{k-1}\Theta_k + \Theta_{k-1}\partial''_k &= 0 \qquad (k > 1) \end{aligned}\right\} \tag{2.17}$$

is called the normal form *of the sequence 2.16.*

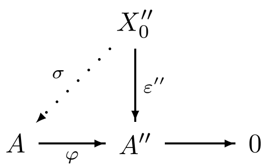
2.18 Proposition. Let $0 \to A' \xrightarrow{\psi} A \xrightarrow{\varphi} A'' \to 0$ be an exact sequence, X' an acyclic positive chain complex augmented over A', and X'' a projective positive chain complex augmented over A''. Then there exists a positive chain complex X augmented over A and homomorphisms Ψ, Φ over ψ and φ respectively such that the sequence

$$0 \longrightarrow X' \xrightarrow{\Psi} X \xrightarrow{\Phi} X'' \longrightarrow 0$$

is exact.

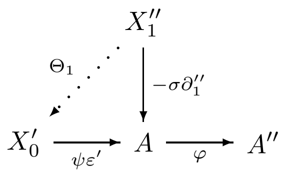
Proof. [Mit65, p. 88] It is enough to find homomorphisms $\sigma\colon X''_0 \to A$ and $\Theta_k\colon X''_k \to X'_{k-1}$ satisfying the conditions 2.17.

Consider the diagram



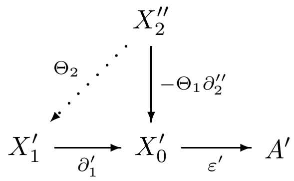
with the row being exact. Since X''_0 is projective by assumption, there exists a homomorphism $\sigma\colon X''_0 \to A$ such that $\sigma\varphi = \varepsilon$.

Next consider the diagram



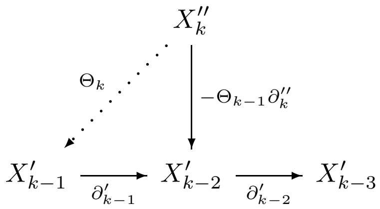
$\operatorname{im}(\psi\varepsilon') = \operatorname{im}(\psi) = \ker(\varphi)$ since ε' is an epimorphism and therefore the row in this diagram is exact. Moreover, X_1'' is projective and $\varphi\sigma\partial_1'' = \varepsilon''\partial_1'' = 0$. Hence one can apply lemma 2.6 which states that there exists a homomorphism $\Theta_1\colon X_1'' \to X_0'$ such that $\psi\varepsilon'\Theta_1 = -\sigma\partial_1''$, that is $\psi\varepsilon'\Theta_1 + \sigma\partial_1'' = 0$.

The homomorphism Θ_2 is defined with the help of the diagram



Since ψ is a monomorphism one can conclude from $-\psi\varepsilon'\Theta_1\partial_2'' = \sigma\partial_1''\partial_2'' = 0$ that $-\varepsilon'\Theta_1\partial_2'' = 0$, too. Moreover, X_2'' is projective and the row of the diagram is exact. Hence one can again apply lemma 2.6, which yields a homomorphism $\Theta_2\colon X_2'' \to X_1'$ such that $\partial_1'\Theta_2 + \Theta_1\partial_2'' = 0$.

Finally, Θ_k $(k > 2)$ is defined inductively by the diagram



with exact row and X_k'' projective. By induction one gets $-\partial_{k-2}'\Theta_{k-1}\partial_k'' = \partial_{k-2}'\partial_{k-1}'\Theta_k = 0$ and therefore by lemma 2.6 there exists a homomorphism $\Theta_k\colon X_k'' \to X_{k-1}'$ such that $\partial_{k-1}'\Theta_k + \Theta_{k-1}\partial_k'' = 0$. □

If in the previous situation the chain complexes X' and X'' are projective resolutions of A' and A'' respectively, the chain complex X will be a projective resolution of A as the next two lemmas will show:

2.19 Lemma. Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence of chain complexes in $\mathcal{A}$. Then if X' and X'' are both projective so is X.

Proof. For each $k \in \mathbb{Z}$ the exact sequence $0 \to X_k' \to X_k \to X_k'' \to 0$ is split because X_k'' is projective. Hence $X_k \cong X_k' \oplus X_k''$ and is therefore projective because both summands are projective. Hence the complex is projective. □

2.20 Lemma. Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence over the sequence $0 \to A' \to A \to A'' \to 0$ and assume that both sequences $X' \to A' \to 0$ and $X'' \to A'' \to 0$ are exact. Then the sequence $X \to A \to 0$ is also exact.

Proof. The homology functor H is a half exact functor (corollary 2.27). Hence the sequence

$$\underbrace{H(X' \to A' \to 0)}_{=0} \to H(X \to A \to 0) \to \underbrace{H(X'' \to A'' \to 0)}_{=0}$$

is exact and necessarily $H(X \to A \to 0) = 0$. □

Combining the results of proposition 2.18, lemma 2.19 and lemma 2.20 yields the

2.21 PROPOSITION. Let $\mathcal{A}$ be an abelian category with projectives. Then for every short exact sequence $0 \to A' \to A \to A'' \to 0$ there exists a projective resolution $0 \to X' \to X \to X'' \to 0$. □

Dually one has

2.22 PROPOSITION. Let $\mathcal{A}$ be an abelian category with injectives. Then for every short exact sequence $0 \to A' \to A \to A'' \to 0$ there exists an injective resolution $0 \to X' \to X \to X'' \to 0$. □

2.3 Derived Functors

For simplicity of notation I consider again the prototype of an additive functor of abelian categories $t : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ as introduced in section 1.1, covariant in the first variable, contravariant in the last variable.

Moreover I assume in this section that the category $\mathcal{A}_1$ has projectives and that the category $\mathcal{A}_2$ has injectives. Therefore it follows from lemma 2.3 that $\mathcal{A}_1$ has projective resolutions and $\mathcal{A}_2$ has injective resolutions.

Let A_1 be an object of $\mathcal{A}_1$ and A_2 an object of $\mathcal{A}_2$. Since $\mathcal{A}_1$ has by assumption projectives there exists a projective resolution P of A_1. Similarly, there exists an injective resolution Q of A_2. Thus $t(P, Q)$ is a positive bi-graded complex in $\mathcal{A}$. Since $t(P, Q)$ is positive, the complex has an associated complex, $\tilde{t}(P, Q)$.

Usually the complex $\tilde{t}(P, Q)$ depends on the choice of P and Q. Let P' and Q' be an other pair of resolutions of A_1 and A_2 respectively. Then by proposition 2.5 and proposition 2.8, $P \simeq P'$ and $Q \simeq Q'$, hence also $t(P, Q) \simeq t(P', Q')$ and finally $\tilde{t}(P, Q) \simeq \tilde{t}(P', Q')$. That is, $\tilde{t}(P, Q)$ is unique up to homotopy and therefore $Ht(P, Q)$ is independent of the choice of the resolutions P and Q.

Assume that $f_1 \colon A_1 \to A_1'$ and $f_2 \colon A_2' \to A_2$ are homomorphisms in $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, P a projective resolution of A_1, P' a projective resolution of A_1', Q an injective resolution of A_2 and Q' an injective resolution of A_2'. Then homomorphisms $F_1 \colon P \to P'$ over f_1 and $F_2 \colon Q' \to Q$ over f_2 exist. Any such homomorphisms are homotopic (proposition 2.5 and proposition 2.8). They define a homomorphism $t(F_1, F_2) \colon t(P, Q) \to t(P', Q')$ over $t(f_1, f_2)$ up to homotopy and hence a unique homomorphism $Ht(F_1, F_2) \colon Ht(P, Q) \to Ht(P', Q')$.

Now the mappings $(A_1, A_2) \mapsto H_i t(P, Q)$ and $(f_1, f_2) \mapsto H_i t(F_1, F_2)$ are functorial ($i \in \mathbb{Z}$), and are non-trivial only for $i \geq 0$. This leads to the important definition of derived functors:

2.23 DEFINITION. *Let $t \colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be an additive functor of abelian categories, covariant in the first and contravariant in the second variable. Assume that the category $\mathcal{A}_1$ has projectives and the category $\mathcal{A}_2$ has injectives. Then the* left i-th derived functor of t ($i \in \mathbb{N}$) *is the functor*

$$t_i \colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$$

which maps $(A_1, A_2) \mapsto H_i t(P, Q)$ *and* $(f_1, f_2) \mapsto H_i t(F_1, F_2)$*, where* P*,* Q*,* F_1 *and* F_2 *are as defined above.*

2.24 REMARK. In the above definition of the (left) derived functors projective resolutions where used to resolve covariant variables and injective resolutions to resolve contravariant variables. If one resolves covariant variables with injective resolutions and contravariant variables with projective resolutions one obtains *right derived functors* t'_i.

Observe that if A_1 is projective, then

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow A_1 \longrightarrow 0$$

is a projective resolution of A_1. Similarly, if A_2 is injective, then

$$0 \longrightarrow A_2 \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

is a injective resolution of A_2. Hence one gets immediately from the definition of the derived functors the

2.25 LEMMA. If A_1 is a projective object or A_2 is an injective object, then $t_i(A_1, A_2) = 0$ for $i > 0$. Similarly, if A_1 is an injective object or A_2 is a projective object, then $t'_i(A_1, A_2) = 0$ for $i > 0$. □

There exists certain long exact sequences involving the derived functors. These sequences are constructed from the long exact homology sequences which are known algebraic homology but exists also in the more general context of abstract homology in abelian categories.

2.26 THEOREM. Let $\mathcal{A}$ be an abelian category and $0 \to X' \to X \to X'' \to 0$ be an exact sequence of chain complexes in $\mathcal{A}$. Then there exists a homomorphism $\delta\colon H(X'') \to H(X')$ of degree -1 and a long exact sequence

$$\ldots \longrightarrow H_{i+1}(X'') \xrightarrow{\delta_{i+1}} H_i(X') \longrightarrow H_i(X) \longrightarrow H_i(X'')$$

$$\xrightarrow{\delta_i} H_{i-1}(X') \longrightarrow H_{i-1}(X) \longrightarrow H_{i-1}(X'') \longrightarrow \ldots$$

□

2.27 COROLLARY. The functor $H\colon \partial\mathcal{A} \to \partial\mathcal{A}$ is a half-exact functor. □

Proof of theorem 2.26. Let X be a chain complex in an abelian category $\mathcal{A}$.

Then since $\operatorname{im}(\partial_{k+1})$ is a subobject of $\ker(\partial_k)$ one has epimorphisms

$$X_k \longrightarrow \operatorname{coker}(\partial_{k+1}) \longrightarrow \operatorname{coim}(\partial_k),$$

inclusions

$$\operatorname{im}(\partial_k) \longrightarrow \ker(\partial_{k-1}) \longrightarrow X_{k-1}$$

and an isomorphism $\operatorname{coim}(\partial_k) \cong \operatorname{im}(\partial_k)$.

Putting these results together one obtains the following factorization of ∂_k

$$X_k \longrightarrow \underbrace{Z'_k \longrightarrow B'_k \longrightarrow B_{k-1} \longrightarrow Z_{k-1}}_{=:\tilde{\partial}_k\colon Z'_k \to Z_{k-1}} \longrightarrow X_{k-1}$$

The composite homomorphism $Z'_k \to B'_k \to B_{k-1} \to Z_{k-1}$ is denoted by $\tilde{\partial}_k$. Since $B'_k \to B_{k-1} \to Z_{k-1}$ is a monomorphism, $\ker(\tilde{\partial}_k) = \ker(Z'_k \to B'_k)$. Similarly, since $Z'_k \to B'_k \to B_{k-1}$ is an epimorphism, $\operatorname{coker}(\tilde{\partial}_k) = \operatorname{coker}(B_{k-1} \to Z_{k-1}) = \ker(\partial_{k-1})/\operatorname{im}(\partial_k)$.

Moreover the sequence

$$0 \longrightarrow \operatorname{im}(\partial_{k+1}) \longrightarrow \ker(\partial_k) \longrightarrow X_k/\operatorname{im}(\partial_{k+1}) \longrightarrow X_k/\ker(\partial_k) \longrightarrow 0$$

is exact and from this one derives the exact sequence

$$0 \longrightarrow \ker(\partial_k)/\operatorname{im}(\partial_{k+1}) \longrightarrow X_k/\operatorname{im}(\partial_{k+1}) \longrightarrow X_k/\ker(\partial_k) \longrightarrow 0$$

The exactness condition then yields the equality

$$\ker(\partial_k)/\operatorname{im}(\partial_{k+1}) = \ker(X_k/\operatorname{im}(\partial_{k+1}) \to X_k/\ker(\partial_k))$$

Putting these pieces together gives

$$\begin{aligned} \operatorname{coker}(\tilde{\partial}_{k+1}) &= \ker(\partial_k)/\operatorname{im}(\partial_{k+1}) \\ &= \ker(X_k/\operatorname{im}(\partial_{k+1}) \to X_k/\ker(\partial_k)) = \ker(\tilde{\partial}_k), \end{aligned}$$

that is

$$\operatorname{coker}(\tilde{\partial}_{k+1}) = H_k(X) = \ker(\tilde{\partial}_k)$$

Now given a short exact sequence $0 \to X' \to X \to X'' \to 0$ of chain complexes on can construct the commutative diagram

$$\begin{array}{ccccccccc}
 & & H_k(X') & \longrightarrow & H_k(X) & \longrightarrow & H_k(X'') & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Z'_k(X') & \longrightarrow & Z'_k(X) & \longrightarrow & Z'_k(X'') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{\partial}'_k} & & \downarrow{\scriptstyle \tilde{\partial}_k} & & \downarrow{\scriptstyle \tilde{\partial}''_k} & & \\
0 & \longrightarrow & Z_{k-1}(X') & \longrightarrow & Z_{k-1}(X) & \longrightarrow & Z_{k-1}(X'') & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H_{k-1}(X') & \longrightarrow & H_{k-1}(X) & \longrightarrow & H_{k-1}(X'') & &
\end{array}$$

with exact rows and columns. By the Snake Lemma B.1 there exists a connecting homomorphism

$$\delta_k \colon H_k(X'') \to H_{k-1}(X')$$

making the sequence

$$H_k(X') \longrightarrow H_k(X) \longrightarrow H_k(X'') \xrightarrow{\delta_k} H_{k-1}(X') \longrightarrow H_{k-1}(X) \longrightarrow H_{k-1}(X'')$$

exact, which proves the claim of theorem 2.26. □

As a straight consequence of theorem 2.26 one gets

2.28 THEOREM. Let $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be an additive functor of abelian categories, covariant in the first and contravariant in the second variable. Assume that the category $\mathcal{A}_1$ has projectives and $\mathcal{A}_2$ has injectives.

If $0 \to A'_k \to A_k \to A''_k \to 0$ $(k = 1, 2)$ are exact sequences in $\mathcal{A}_k$ then there exist long exact sequences

$$\ldots \longrightarrow t_2(A''_1, A_2) \longrightarrow t_1(A'_1, A_2) \longrightarrow t_1(A_1, A_2) \longrightarrow t_1(A''_1, A_2)$$
$$\longrightarrow t_0(A'_1, A_2) \longrightarrow t_0(A_1, A_2) \longrightarrow t_0(A''_1, A_2) \longrightarrow 0$$

and

$$\ldots \longrightarrow t_2(A_1, A'_2) \longrightarrow t_1(A_1, A''_2) \longrightarrow t_1(A_1, A_2) \longrightarrow t_1(A_1, A'_2)$$
$$\longrightarrow t_0(A_1, A''_2) \longrightarrow t_0(A_1, A_2) \longrightarrow t_0(A_1, A'_2) \longrightarrow 0$$

Proof. The exactness of the sequences follows from the fact that these are homology sequences of suitable exact sequences. These sequences exist due to proposition 2.21 and proposition 2.22. □

2.29 COROLLARY. The derived functors t_i are half exact functors. The functor t_0 is a right exact functor. t_0 is exact if and only if $t_1 = 0$. □

If P is a projective resolution of A_1 and Q an injective resolution of A_2, then the augmentation homomorphisms $\varepsilon_1\colon P \to A_1$ and $\varepsilon_2 : A_2 \to Q$ yield a homomorphism $t(P, Q) \to t(A_1, A_2)$. This defines a natural map

$$\sigma_0 : t_0 \to t$$

2.30 PROPOSITION. The natural map σ_0 is an equivalence if and only if the functor t is right exact.

Proof. If σ_0 is an equivalence t is right exact since t_0 is (corollary 2.29). On the other hand, if t is right exact, the sequence

$$t(P_1, Q_0) \oplus t(P_0, Q_{-1}) \xrightarrow{\psi} t(P_0, Q_0) \xrightarrow{\varphi} t(A_1, A_2) \to 0$$

is exact by lemma 1.12. But now, since φ is an epimorphism, $t(A_1, A_2) \cong t(P_0, Q_0)/\ker(\varphi) = t(P_0, Q_0)/\operatorname{im}(\psi) \cong H_0 t(P, Q) = t_0(A_1, A_2)$. Hence the map σ_0 is an equivalence. □

2.31 COROLLARY. Let $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}$ and t be as in theorem 2.28. Assume that $0 \to A'_k \to A_k \to A''_k \to 0$ $(k = 1, 2)$ are exact sequences in $\mathcal{A}_k$. If t is right exact then there exist long exact sequences

$$\ldots \longrightarrow t_2(A''_1, A_2) \longrightarrow t_1(A'_1, A_2) \longrightarrow t_1(A_1, A_2) \longrightarrow t_1(A''_1, A_2)$$
$$\longrightarrow t(A'_1, A_2) \longrightarrow t(A_1, A_2) \longrightarrow t(A''_1, A_2) \longrightarrow 0$$

and

$$\ldots \longrightarrow t_2(A_1, A_2') \longrightarrow t_1(A_1, A_2'') \longrightarrow t_1(A_1, A_2) \longrightarrow t_1(A_1, A_2')$$
$$\longrightarrow t(A_1, A_2'') \longrightarrow t(A_1, A_2) \longrightarrow t(A_1, A_2') \longrightarrow 0$$

□

Dually, for the right derived functors t_i' the analogous theorem to theorem 2.28 reads:

2.32 Theorem. Let $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be an additive functor of abelian categories, covariant in the first and contravariant in the second variable. Assume that the category $\mathcal{A}_1$ has injectives and $\mathcal{A}_2$ has projectives.

If $0 \to A_k' \to A_k \to A_k'' \to 0$ $(k = 1, 2)$ are exact sequences in $\mathcal{A}_k$, then there exist long exact sequences

$$0 \longrightarrow t_0'(A_1', A_2) \longrightarrow t_0'(A_1, A_2) \longrightarrow t_0'(A_1'', A_2) \longrightarrow t_1'(A_1', A_2)$$
$$\longrightarrow t_1'(A_1, A_2) \longrightarrow t_1'(A_1'', A_2) \longrightarrow t_2'(A_1', A_2) \longrightarrow \ldots$$

and

$$0 \longrightarrow t_0'(A_1, A_2'') \longrightarrow t_0'(A_1, A_2) \longrightarrow t_0'(A_1, A_2') \longrightarrow t_1'(A_1, A_2'')$$
$$\longrightarrow t_1'(A_1, A_2) \longrightarrow t_1'(A_1, A_2') \longrightarrow t_2'(A_1, A_2'') \longrightarrow \ldots$$

□

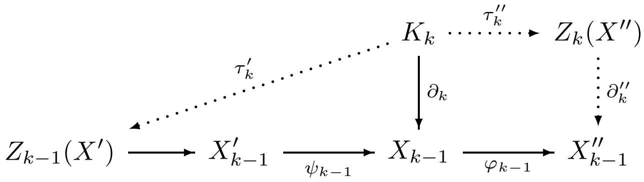
Now one can define in a similar fashion as for left derived functors a natural map $\tau_0\colon t \to t_0'$ [CE56, p. 89] and obtains in an analogous way the dual to proposition 2.30, which reads

2.33 Proposition. The natural map τ_0 is an equivalence if and only if the functor t is left exact. □

Applying this to theorem 2.32 one derives the dual to corollary 2.31.

For later use (see proof of proposition 3.8) an alternative description of the connecting homomorphism $\delta_k\colon H_k(X'') \to H_{k-1}(X')$ is the following:

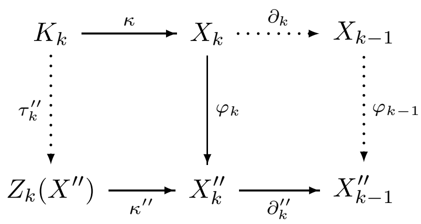
Consider the diagram



(2.34)

where K_k is the kernel of the composite $\varphi_{k-1} \circ \partial_k$.

- To define the homomorphism τ_k'' consider the diagram



where the homomorphism κ is the kernel of the composite $\varphi_{k-1} \circ \partial_k = \partial''_k \circ \varphi_k$ and the homomorphism κ'' is the kernel of ∂''_k.

Then by proposition 13.2 of [Mit65, p. 15] this diagram can be extended by $\tau''_k \colon K_k \to Z_k(X'')$ to a pullback diagram. Now since $\mathcal{A}$ is an abelian category and φ_k is by assumption an epimorphism, τ''_k is an epimorphism, too (proposition 20.2 in [Mit65, p. 34]).

- On the other hand $\partial_k(K_k) \subset \operatorname{im}(\psi_{k-1}) = \ker(\phi_{k-1})$ and $\partial_k(K_k) \subset \ker(\partial_{k-1})$. Since ψ_{k-1} is a monomorphism, ψ_{k-1} induces a isomorphism $\psi^{-1}_{k-1} \colon \operatorname{im}(\psi_{k-1}) \to X'_{k-1}$. Hence one can define a homomorphism $\tau'_k := \psi^{-1}_{k-1}\partial_k \colon K_k \to X'_{k-1}$. Now $\partial'_{k-1}\tau'_k(K_k) = \psi^{-1}_{k-2}\partial_{k-1}\partial_k(K_k) = 0$, that is $\tau'_k(K_k) \subset \ker(\partial'_{k-1}) = Z_{k-1}(X')$. Hence τ'_k can also be seen as a homomorphism $\tau'_k \colon K_k \to Z_{k-1}(X')$.

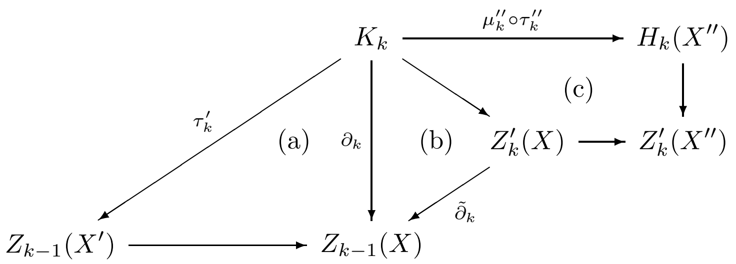
Denote by $\mu''_k \colon Z_k(X'') \to H_k(X'')$ and $\mu'_{k-1} \colon Z_{k-1}(X') \to H_{k-1}(X')$ he canonical projections.

2.35 Proposition. Using the above notation one has

$$\mu'_{k-1} \circ \tau'_k = \delta_k \circ \mu''_k \circ \tau''_k$$

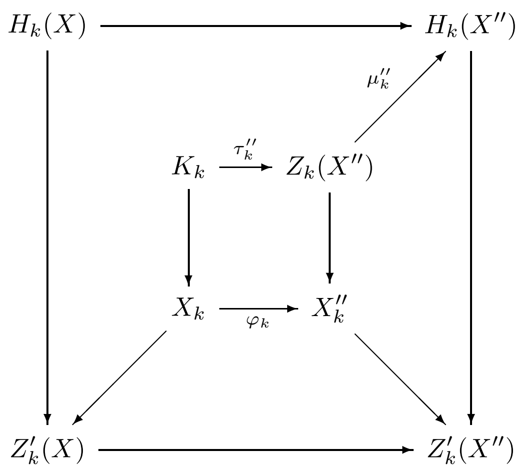
where the homomorphism $\delta_k \colon H_k(X'') \to H_{k-1}(X')$ is the connection homomorphism as defined in theorem 2.26.

Proof. Consider the diagram



(2.36)

In this diagram the triangles (a) and (b) are commutative due to the definitions of the involved homomorphisms. The commutativity of the rectangle (c) follows from the commutativity of the diagram



With these observations one can now start to chase members in diagram 2.36 (as described in appendix A) to prove the claim. The description of the effect of the connecting homomorphism on a member of $H_k(X'')$ as described in the proof of the Snake Lemma B.1 is used.

Let $x \in_m K_k$. It has to be shown that $\delta_k \circ \mu''_k \circ \tau''_k \circ x \equiv \mu'_{k-1} \circ \tau'_k \circ x$. Therefore denote $x'' := \mu''_k \circ \tau''_k \circ x \in_m H_k(X'')$ and observe that the effect of δ_k on x'' is as follows:

Set $z'' := (H_k(X'') \to Z'_k(X'')) \circ x''$. Since $Z'_k(X) \to Z'_k(X'')$ is an epimorphism there exists a $z \in_m Z'_k$ such that $(Z'_k(X) \to Z'_k(X'')) \circ z \equiv z''$. Since $\delta_k \circ x''$ does not depend on the choice of z and since the rectangle (c) in diagram 2.36 is commutative one can choose z to be $(K_k \to Z'(X)) \circ x$. Then $\tilde{\partial}_k \circ z \in_m Z'_{k-1}$. Now $\tilde{\partial}_k \circ z \in_m \ker(Z_{k-1}(X) \to Z_{k-1}(X'')) = \operatorname{im}(Z_{k-1}(X') \to Z_{k-1}(X'))$ and hence there exists a $z' \in_m Z_{k-1}(X')$ such that $(Z_{k-1}(X') \to Z_{k-1}(X')) \circ z' \equiv \tilde{\partial}_k \circ z$. Since $\delta_k \circ x''$ does not depend on the choice of z' and due to the commutativity of the triangles (a) and (b) in diagram 2.36, one can assume $z' \equiv \tau'_k \circ x$. Therefore $\mu'_{k-1} \circ \tau'_k \circ x \equiv \mu'_{k-1} \circ z' \equiv \delta_k \circ \mu''_k \circ \tau''_k \circ x$ as had to be shown and from which the claim follows immediately. $\square$

3 The Künneth Formula

Given a right exact additive functor $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ we will state for chain complexes X_1 in $\mathcal{A}_1$ and X_2 in $\mathcal{A}_2$ the existence of a certain homomorphism $\alpha\colon t(H(X_1), H(X_2)) \to Ht(X_1, X_2)$. The Künneth Formula will then state in the case of a right exact functor and under certain conditions the existence of a homomorphism β such that the sequence

$$0 \longrightarrow t(H(X_1), H(X_2)) \xrightarrow{\alpha} Ht(X_1, X_2) \xrightarrow{\beta} t_1(H(X_1), H(X_2)) \longrightarrow 0$$

is exact. Finally more conditions will then imply that this sequence splits. A dual result exist for left exact functors.

Many of the general results will depend only slightly on the type of the functor (right or left exact) and the variance of the variable used. Hence the results will be proven only for one case.

3.1 The Homomorphism α

Let $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be an additive functor, covariant in the first variable and contravariant in the second variable.

Let X_1 be a chain complex in $\mathcal{A}_1$ and X_2 a chain complex in $\mathcal{A}_2$. Then one has the commutative diagrams

$$\begin{array}{ccc} Z(X_1) & \longrightarrow & H(X_1) \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & Z'(X_1) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Z'(X_2) & \longleftarrow & H(X_2) \\ \uparrow & & \uparrow \\ X_2 & \longleftarrow & Z(X_2) \end{array}$$

Applying the functor t yields the commutative diagram

$$\begin{array}{ccc} t(Z(X_1), Z'(X_2)) & \longrightarrow & t(H(X_1), H(X_2)) \\ \downarrow & & \downarrow \\ t(X_1, X_2) & \longrightarrow & t(Z'(X_1), Z(X_2)) \end{array}$$

where all complexes are seen as single graded complexes. Now $t(Z(X_1), Z'(X_2))$, $t(H(X_1), H(X_2))$ and $t(Z'(X_1), Z(X_2))$ have trivial differentiations, hence they are isomorphic to their homology sequences. Therefore applying the homology functor to the diagram yields

$$\begin{array}{ccc} t(Z(X_1), Z'(X_2)) & \xrightarrow{\xi} & t(H(X_1), H(X_2)) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\tau} \\ Ht(X_1, X_2) & \xrightarrow[\zeta]{} & t(Z'(X_1), Z(X_2)) \end{array} \tag{3.1}$$

3.2 Theorem. If the above functor $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ is right exact then there exists a unique homomorphism

$$\alpha\colon t(H(X_1), H(X_2)) \to Ht(X_1, X_2)$$

of degree 0 preserving the commutativity of diagram 3.1. This homomorphism is natural relative to chain maps $X_1 \to X_1'$ and $X_2' \to X_2$. If X_1 and X_2 have trivial differentiations then α is the identity. The last two properties characterize the homomorphism α uniquely.

Proof. [CE56, p. 65] One has the exact sequences $Z(X_1) \to H(X_1) \to 0$ and $0 \to H(X_2) \to Z'(X_2)$. Since the functor t is right exact, the sequences

$$t(Z(X_1), Z'(X_2)) \to t(H(X_1), Z'(X_2)) \to 0$$

and

$$t(H(X_1), Z'(X_2)) \to t(H(X_1), H(X_2)) \to 0$$

are exact, and hence the composition $\xi\colon t(Z(X_1), Z'(X_2)) \to t(H(X_1), H(X_2))$ is an epimorphism. Hence there can exist at most one homomorphism α such that $\alpha\xi = \eta$.

To prove the existence of the homomorphism α it is enough to show that $\ker(\xi) \subset \ker(\eta)$.

Applying lemma 1.12 to the exact sequences $B(X_1) \to Z(X_1) \to H(X_1) \to 0$ and $0 \to H(X_2) \to Z'(X_2) \to B'(X_2)$ yields the exact sequence

$$\begin{aligned} t(B(X_1), Z'(X_2)) \oplus t(Z(X_1), B'(X_2)) \xrightarrow{\varphi} t(Z(X_1), Z'(X_2)) \\ \xrightarrow{\xi} t(H(X_1), H(X_2)) \longrightarrow 0 \end{aligned}$$

Hence the kernel $\ker(\xi)$ is the direct sum of the images of

$$t(B(X_1), Z'(X_2)) \longrightarrow t(Z(X_1), Z'(X_2))$$

and

$$t(Z(X_1), B'(X_2)) \longrightarrow t(Z(X_1), Z'(X_2))$$

Hence to show that $\ker(\xi) \subset \ker(\eta)$ it suffices to show that the composite homomorphisms

$$t(B(X_1), Z'(X_2)) \longrightarrow t(Z(X_1), Z'(X_2)) \xrightarrow{\eta} Ht(X_1, X_2) \tag{3.3}$$

and

$$t(Z(X_1), B'(X_2)) \longrightarrow t(Z(X_1), Z'(X_2)) \xrightarrow{\eta} Ht(X_1, X_2) \tag{3.4}$$

are both zero. Since the homomorphism η admits the factorizations

$$t(Z(X_1), Z'(X_2)) \longrightarrow Ht(X_1, Z'(X_2)) \longrightarrow Ht(X_1, X_2)$$

and

$$t(Z(X_1), Z'(X_2)) \longrightarrow Ht(Z(X_1), X_2) \longrightarrow Ht(X_1, X_2)$$

the homomorphisms 3.3 and 3.4 admit the factorizations

$$t(B(X_1), Z'(X_2)) \xrightarrow{\beta} Ht(X_1, Z'(X_2)) \longrightarrow Ht(X_1, X_2)$$

and

$$t(Z(X_1), B'(X_2)) \xrightarrow{\gamma} Ht(Z(X_1), X_2) \longrightarrow Ht(X_1, X_2)$$

respectively and it is sufficient to show that the homomorphisms β and γ are zero.

Therefore consider the factorization

$$t(X_1, Z'(X_2)) \longrightarrow t(B'(X_1), Z'(X_2)) \xrightarrow{t(\tilde{\partial}, \mathrm{id})} t(B(X_1), Z'(X_2)) \xrightarrow{\beta'} t(X_1, Z'(X_2))$$

of the differential operator $t(\partial, \mathrm{id})\colon t(X_1, Z'(X_2)) \to t(X_1, Z'(X_2))$. Since t is right exact the homomorphism $t(X_1, Z'(X_2)) \to t(B'(X_1), Z'(X_2))$ is an epimorphism. Since $\tilde{\partial}$ is the isomorphism induced by $\partial\colon X_1 \to X_1$ the homomorphism $t(\tilde{\partial}, \mathrm{id})$ is an isomorphism as well. Hence $\mathrm{im}(\beta') = \mathrm{im}(t(\partial, \mathrm{id})) = Bt(X_1, Z'(X_2))$. Now observe that the homomorphism β is induced by β', which leads to $\mathrm{im}(\beta)$ being the zero.

Similarly one concludes from the factorization

$$t(Z(X_1), X_2) \longrightarrow t(Z(X_1), B(X_2)) \xrightarrow{t(\mathrm{id}, \tilde{\partial})} t(Z(X_1), B'(X_2)) \xrightarrow{\beta'} t(Z(X_1), X_2)$$

of the differential operator $t(\mathrm{id}, \partial)\colon t(Z(X_1), X_2) \to t(Z(X_1), X_2)$ that $\mathrm{im}(\gamma)$ is zero.

The naturality of α and the fact that α is the identity if X_1 and X_2 have trivial differentiations are obvious.

Consider the chain complexes $Z(X_1)$ and $Z'(X_2)$. Since these complexes have trivial differentiation they are equal to their homology, that is $Z(X_1) = HZ(X_1)$ and $Z'(X_2) = HZ'(X_2)$. Then the canonical homomorphisms $Z(X_1) \to X_1$ and $X_2 \to Z'(X_2)$ induce an epimorphism $Z(X_1) = HZ(X_1) \to H(X_1)$ and a monomorphism $H(X_2) \to HZ'(X_2) = Z'(X_2)$.

Now with these homomorphisms in mind consider the commutative diagram

$$\begin{array}{ccc}
t(Z(X_1), Z'(X_2)) = t(HZ(X_1), HZ'(X_2)) & \xrightarrow{\xi} & t(H(X_1), H(X_2)) \\
\Big\downarrow \mathrm{id} & & \Big\downarrow \alpha \\
t(Z(X_1), Z'(X_2)) = Ht(Z(X_1), Z'(X_2)) & \xrightarrow{\eta} & Ht(X_1, X_2)
\end{array}$$

where ξ is the epimorphism $t(Z(X_1) \to H(X_1), H(X_2) \to Z'(X_2))$.

If $\alpha'\colon t(H(X_1), H(X_2)) \to Ht(X_1, X_2)$ is another homomorphism satisfying the two conditions (naturality and identity) then the commutativity is preserved, if inserted into the above diagram. Hence $\alpha' \circ \xi = \eta = \alpha \circ \xi$. Since ξ is an epimorphism it follows that $\alpha' = \alpha$, that is the two conditions characterize the homomorphism α uniquely. □

If the functor t is left exact there exists a similar homomorphism, however it is in the opposite direction:

3.5 Theorem. If the functor $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ is left exact then there exists a unique homomorphism

$$\alpha'\colon Ht(X_1, X_2) \to t(H(X_1), H(X_2))$$

of degree 0 preserving the commutativity of diagram 3.1. This homomorphism is natural relative to chain maps $X_1 \to X_1'$ and $X_2' \to X_2$. If X_1 and X_2 have trivial differentiations then α' is the identity. The last two properties characterize the homomorphism α' uniquely. □

The homomorphisms α and α' commute under certain conditions with connecting homomorphisms as follows:

3.6 Proposition. Let $0 \to X' \xrightarrow{\psi} X \xrightarrow{\varphi} X'' \to 0$ be a short exact sequence of chain complexes in $\mathcal{A}_1$ and Y a chain complex in $\mathcal{A}_2$ such that the sequence

$$0 \to t(X', Y) \to t(X, Y) \to t(X'', Y) \to 0$$

is exact. Let $\delta\colon H(X'') \to H(X')$ and $\Delta\colon Ht(X'', Y) \to Ht(X', Y)$ be the connecting homomorphisms. Then if the functor t is right exact the diagram

$$\begin{array}{ccc} t(H(X''), H(Y)) & \xrightarrow{t(\delta,\mathrm{id})} & t(H(X'), H(Y)) \\ \Big\downarrow{\scriptstyle \alpha_2} & & \Big\downarrow{\scriptstyle \alpha_1} \\ Ht(X'', Y) & \xrightarrow[\Delta]{} & Ht(X', Y) \end{array}$$

is commutative.

3.7 Remark. For a left exact functor t this proposition reads the same except that the homomorphisms α_1, α_2 are replaced by α_1', α_2' respectively, which point in the opposite direction. [CE56, p. 67]

One can state this proposition also for the contravariant variable. It reads

3.8 Proposition. Let $0 \to Y' \xrightarrow{\psi} Y \xrightarrow{\varphi} Y'' \to 0$ be a short exact sequence of chain complexes in $\mathcal{A}_2$ and X a chain complex in $\mathcal{A}_1$ such that the sequence

$$0 \to t(X, Y'') \to t(X, Y) \to t(X, Y') \to 0$$

is exact. Let $\delta\colon H(Y'') \to H(Y')$ and $\Delta\colon Ht(X, Y') \to Ht(X, Y'')$ be the connecting homomorphisms. Then if the functor t is right exact the diagram

$$\begin{array}{ccc} t(H(X), H(Y')) & \xrightarrow{t(\mathrm{id},\delta)} & t(H(X), H(Y'')) \\ \Big\downarrow{\scriptstyle \alpha_2} & & \Big\downarrow{\scriptstyle \alpha_1} \\ Ht(X, Y') & \xrightarrow[\Delta]{} & Ht(X, Y'') \end{array}$$

is commutative. □

3.9 Remark. Here the homomorphism $t(\mathrm{id}, \delta)$ involves a sign, see Section 1.3.

Yet again, for a left exact functor t this proposition reads the same except that the homomorphisms α_1, α_2 are replaced by α_1', α_2' respectively, which point in the opposite direction. [CE56, p. 67]

Proof of proposition 3.6. [CE56, pp. 67ff.] Let K_k be the kernel of the composition $\varphi_{k-1} \circ \partial_k \colon X_k \to X''_{k-1}$. Then by proposition 2.35 there exists homomorphisms

$$H_{k-1}(X') \xleftarrow{\mu'_{k-1}} Z_{k-1}(X') \xleftarrow{\tau'_k} K_k \xrightarrow{\tau''_k} Z_k(X'') \xrightarrow{\mu''_k} H_k(X'')$$

such that

$$\mu'_{k-1} \circ \tau'_k = \delta_k \circ \mu''_k \circ \tau''_k \tag{3.10}$$

Similarly let M_k be the kernel of the composition

$$t(X,Y)_k \xrightarrow{\partial_k} t(X,Y)_{k-1} \longrightarrow t(X'',Y)_{k-1} \tag{3.11}$$

Then again by proposition 2.35 there are homomorphisms

$$H_{k-1}t(X',Y) \xleftarrow{\rho'_{k-1}} Z_{k-1}t(X',Y) \xleftarrow{\sigma'_k} M_k \xrightarrow{\sigma''_k} Z_k t(X'',Y) \xrightarrow{\rho''_k} H_k t(X'',Y)$$

such that

$$\rho'_{k-1} \circ \sigma'_k = \Delta_k \circ \rho''_k \circ \sigma''_k \tag{3.12}$$

Composing the map $t(K, Z'(Y))_k \to t(X,Y)_k$ with the homomorphism 3.11 yields the zero homomorphism. Hence this map factors through the kernel M_k, that is, there exists a homomorphism

$$\Theta_k \colon t(K, Z'(Y))_k \to M_k$$

Now consider the diagrams

$$\begin{array}{ccccc} t(H(X'), H(Y)) & \xleftarrow{t(\mu', i)} & t(Z(X'), Z'(Y)) & \xleftarrow{t(\tau', \mathrm{id})} & t(K, Z'(Y)) \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow & & \downarrow{\scriptstyle \Theta} \\ Ht(X', Y) & \xleftarrow[\rho']{} & Zt(X', Y) & \xleftarrow[\sigma']{} & M \end{array} \tag{3.13}$$

and

$$\begin{array}{ccccc} t(K, Z'(Y)) & \xrightarrow{t(\tau'', \mathrm{id})} & t(Z(X''), Z'(Y)) & \xrightarrow{t(\mu'', i)} & t(H(X''), H(Y)) \\ \downarrow{\scriptstyle \Theta} & & \downarrow & & \downarrow{\scriptstyle \alpha_2} \\ M & \xrightarrow[\sigma'']{} & Zt(X'', Y) & \xrightarrow[\rho'']{} & Ht(X'', Y) \end{array} \tag{3.14}$$

where i is the inclusion $i \colon H(Y) \to Z'(Y)$.

The commutativity of the left rectangle in diagram 3.13 and the right rectangle in diagram 3.14 follows straight from the definition of the the homomorphism α_1 and α_2.

The commutativity of the remaining two rectangles follows easily from the definition of the homomorphism Θ. [CE56, p. 68]

With this and equations 3.10 and 3.12 it follows that

$$\begin{aligned}\Delta \circ \alpha_2 \circ t(\mu''\tau'', i) &= \Delta \circ \rho'' \circ \sigma'' \circ \Theta = \rho' \circ \sigma' \circ \Theta = \\ &= \alpha_1 \circ t(\mu'\tau', i) = \alpha_1 \circ t(\delta\mu''\tau'', i) = \\ &= \alpha_1 \circ t(\delta, \mathrm{id}) \circ t(\mu''\tau'', i)\end{aligned}$$

Since μ'', τ'' are epimorphisms, i is a monomorphism and t is a right exact functor, the homomorphism $t(\mu''\tau'', i)$ is necessarily an epimorphism. Hence $\Delta \circ \alpha_2 = \alpha_1 \circ t(\delta, \mathrm{id})$. □

3.15 Proposition. Let $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be a right exact functor, covariant in the first variable and contravariant in the second variable. Let X_1 be a chain complex in $\mathcal{A}_1$ and X_2 a chain complex in $\mathcal{A}_2$. Assume that the the sequence

$$0 \longrightarrow t(Z(X_1), X_2) \longrightarrow t(X_1, X_2)$$

is exact and that the homomorphism

$$\alpha\colon t(B(X_1), H(X_2)) \longrightarrow Ht(B(X_1), X_2)$$

is an epimorphism. Then the sequence

$$\begin{aligned}\ldots \longrightarrow Ht(X_1, X_2) &\xrightarrow{k_*} Ht(B(X_1), X_2) \\ &\xrightarrow{i_*} Ht(Z(X_1), X_2) \xrightarrow{j_*} Ht(X_1, X_2) \longrightarrow \ldots\end{aligned}$$

induced by the natural homomorphisms

$$X_1 \xrightarrow{k} B(X_1) \xrightarrow{i} Z(X_1) \xrightarrow{j} X_1 \tag{3.16}$$

is exact.

3.17 Remark. The similar result with respect to the second, contravariant variable is obtained (beside the obvious changes) by replacing $Z(X_1)$ and $B(X_1)$ with $Z'(X_2)$ and $B'(X_2)$ respectively and reversing the arrows in 3.16. [CE56, p. 69]

Proof of proposition 3.15. [CE56, p. 70] Since t is a right exact functor and since by assumption the sequence $0 \to t(Z(X_1), X_2) \to t(X_1, X_2)$ is exact, one obtains the short exact sequence

$$0 \longrightarrow t(Z(X_1), X_2) \longrightarrow t(X_1, X_2) \longrightarrow t(B(X_1), X_2) \longrightarrow 0$$

of chain complexes. By theorem 2.26 there exists a long exact sequence

$$\begin{aligned}\ldots \longrightarrow Ht(X_1, X_2) &\xrightarrow{k_*} Ht(B(X_1), X_2) \\ &\xrightarrow{\Delta} Ht(Z(X_1), X_2) \xrightarrow{j_*} Ht(X_1, X_2) \longrightarrow \ldots\end{aligned}$$

Hence it remains to be shown that the homomorphism Δ is equal to i_*.

By proposition 3.6 there exist a commutative diagram

$$\begin{array}{ccc} t(B(X_1), H(X_2)) & \xrightarrow{t(\delta,\mathrm{id})} & t(Z(X_1), H(X_2)) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ Ht(B(X_1), X_2) & \xrightarrow[\Delta]{} & Ht(Z(X_1), X_2) \end{array}$$

where $\delta\colon B(X_1) \to Z(X_1)$ is the connecting homomorphism of the exact sequence $0 \to Z(X_1) \to X_1 \to B(X_1) \to 0$. Now $\delta = i$ and by the naturality of the homomorphism α one concludes that $i_* \circ \alpha = \alpha' \circ t(\delta, \mathrm{id})$ and hence $i_* \circ \alpha = \Delta \circ \alpha$. Since by assumption α is an epimorphism this means that $i_* = \Delta$. □

The dual proposition to proposition 3.15 for left exact functors reads:

3.18 Proposition. Let $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be a left exact functor, covariant in the first variable and contravariant in the second variable. Let X_1 be a chain complex in $\mathcal{A}_1$ and X_2 a chain complex in $\mathcal{A}_2$. Assume that the the sequence

$$t(Z'(X_1), X_2) \longrightarrow t(X_1, X_2) \longrightarrow 0$$

is exact and that the homomorphism

$$\alpha'\colon Ht(B'(X_1), X_2) \longrightarrow t(B'(X_1), H(X_2))$$

is a monomorphism. Then the sequence

$$\ldots \longrightarrow Ht(X_1, X_2) \xrightarrow{k_*} Ht(Z'(X_1), X_2)$$
$$\xrightarrow{i_*} Ht(B'(X_1), X_2) \xrightarrow{j_*} Ht(X_1, X_2) \longrightarrow \ldots$$

induced by the natural homomorphisms

$$X_1 \xrightarrow{k} Z'(X_1) \xrightarrow{i} B'(X_1) \xrightarrow{j} X_1$$

is exact. □

Again, the changes to be done to this proposition if one is interested in the result with respect to the second, contravariant variable are obvious (analogous to Remark 3.17; see also [CE56, p. 70]).

3.2 The Künneth Formula

Now all the tools are prepared to state and prove the Künneth Formula. There are two variants of this theorem, one for right exact functors and a similar one for left exact functors. The theorem will be stated for both cases, but only proven for the case of right exact functors as there is no significant difference in the proof of the dual case.

3.19 Theorem (Künneth Formula for right exact functors). Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}$ be abelian categories and $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ an additive right exact functor, covariant in the first and contravariant in the second variable. Assume that $\mathcal{A}_1$ has projectives, $\mathcal{A}_2$ has injectives and $\mathcal{A}$ has coproducts.

Let X_1 be a chain complex in $\mathcal{A}_1$ and X_2 a chain complex in $\mathcal{A}_2$.

1. If the homomorphisms

$$\begin{aligned}\alpha_1\colon t(B(X_1), H(X_2)) &\to Ht(B(X_1), X_2)\\ \alpha_2\colon t(Z(X_1), H(X_2)) &\to Ht(Z(X_1), X_2)\end{aligned}$$

are isomorphisms, and if

$$t_1(B(X_1), X_2) = 0 = t_1(Z(X_1), H(X_2))$$

or

2. if the homomorphisms

$$\begin{aligned}\alpha_1\colon t(H(X_1), B'(X_2)) &\to Ht(X_1, B'(X_2))\\ \alpha_2\colon t(H(X_1), Z'(X_2)) &\to Ht(X_1, Z'(X_2))\end{aligned}$$

are isomorphisms, and if

$$t_1(X_1, B'(X_2)) = 0 = t_1(H(X_1), Z'(X_2))$$

then there exists a homomorphism β of degree -1 such that the sequence

$$0 \longrightarrow t(H(X_1), H(X_2)) \xrightarrow{\alpha} Ht(X_1, X_2) \xrightarrow{\beta} t_1(H(X_1), H(X_2)) \longrightarrow 0 \quad (3.20)$$

is exact.

In the case of left exact functors the direction of arrows changes, the functors Z and B get exchanged with Z' and B' respectively, and the homomorphisms α and α_k are replaced by the homomorphisms α' and α'_k respectively. Moreover now left derived functors t'_i are used instead of right derived functors t_i:

3.21 Theorem (Künneth Formula for left exact functors). Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}$ be abelian categories and $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ an additive left exact functor, covariant in the first and contravariant in the second variable. Assume that $\mathcal{A}_1$ has injectives, $\mathcal{A}_2$ has projectives and $\mathcal{A}$ has coproducts.

Let X_1 be a chain complex in $\mathcal{A}_1$ and X_2 a chain complex in $\mathcal{A}_2$.

1. If the homomorphisms

$$\begin{aligned}\alpha'_1\colon Ht(B'(X_1), X_2) &\to t(B'(X_1), H(X_2))\\ \alpha'_2\colon Ht(Z'(X_1), X_2) &\to t(Z'(X_1), H(X_2))\end{aligned}$$

are isomorphisms, and if

$$t'_1(B'(X_1), X_2) = 0 = t'_1(Z'(X_1), H(X_2))$$

or

2. if the homomorphisms

$$\alpha_1' \colon Ht(X_1, B(X_2)) \to t(H(X_1), B(X_2))$$
$$\alpha_2' \colon Ht(X_1, Z(X_2)) \to t(H(X_1), Z(X_2))$$

are isomorphisms, and if

$$t_1'(X_1, B(X_2)) = 0 = t_1'(H(X_1), Z(X_2))$$

then there exists a homomorphism β' of degree -1 such that the sequence

$$0 \longrightarrow t_1'(H(X_1), H(X_2)) \xrightarrow{\beta'} Ht(X_1, X_2) \xrightarrow{\alpha'} t(H(X_1), H(X_2)) \longrightarrow 0 \tag{3.22}$$

is exact.

Proof of theorem 3.19. [CE56, p. 71] Assume conditions 1 and consider the commutative diagram

$$\begin{array}{ccccc}
t(B(X_1), H(X_2)) & \xrightarrow{\alpha_1} & Ht(B(X_1), X_2) & & \\
\downarrow & & \downarrow & & \\
t(Z(X_1), H(X_2)) & \xrightarrow{\alpha_2} & Ht(Z(X_1), X_2) & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
t(H(X_1), H(X_2)) & \xrightarrow{\alpha} & Ht(X_1, X_2) & \overset{\beta}{\cdots\cdots\to} & t_1(H(X_1), H(X_2)) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & Ht(B(X_1), X_2) & \xrightarrow{\alpha_1^{-1}} & t(B(X_1), H(X_2)) \\
 & & \downarrow & & \downarrow \\
 & & Ht(Z(X_1), X_2) & \xrightarrow{\alpha_2^{-1}} & t(Z(X_1), H(X_2))
\end{array} \tag{3.23}$$

Since the functor t is right exact the first column of diagram 3.23 is exact.

Applying corollary 2.31 to the short exact sequence $0 \to Z(X_1) \to X_1 \to B(X_1) \to 0$ gives the exact sequence

$$t_1(B(X_1), X_2) \longrightarrow t(Z(X_1), X_2) \longrightarrow t(X_1, X_2)$$

Now by assumption $t_1(B(X_1), X_2) = 0$ and hence one sees that the homomorphism $t(Z(X_1), X_2) \to t(X_1, X_2)$ is a monomorphism. Proposition 3.15 states then that the middle column in diagram 3.23 is exact.

Similarly, applying corollary 2.31 to the short exact sequence $0 \to B(X_1) \to Z(X_1) \to H(X_1) \to 0$ yields the exact sequence

$$t_1(Z(X_1), H(X_2)) \longrightarrow t_1(H(X_1), H(X_2))$$
$$\longrightarrow t(B(X_1), H(X_2)) \longrightarrow t(Z(X_1), H(X_2))$$

Since by assumption $t_1(Z(X_1), H(X_2)) = 0$, the third column of diagram 3.23 is exact.

Now it follows easily that there exists a unique homomorphism

$$\beta\colon Ht(X_1, X_2) \to t_1(H(X_1), H(X_2))$$

which when inserted into diagram 3.23 yields a commutative diagram. Observe that the homomorphism $Ht(X_1, X_2) \to Ht(B(X_1), X_2)$ is the only one which has degree -1, the others have degree zero. Hence β has to have degree -1, too.

The exactness of sequence 3.20 follows straight by chasing members in the diagram 3.23.

The proof is similar under the conditions 2. □

3.24 Remark. Conditions 1 and 2 in theorem 3.19 yield very similar-looking exact sequences. Indeed, in both cases the homomorphism α is the same, but an other question is, whether in both cases the homomorphism β is the same, too. [CE56, p. 113] A similar consideration holds for theorem 3.22.

3.25 Proposition. Let $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be a right exact functor, covariant in the first and contravariant in the second variable, and let X_1 be a chain complex in $\mathcal{A}_1$ and X_2 a chain complex in $\mathcal{A}_2$. Assume that the exact sequences

$$0 \longrightarrow H(X_1) \longrightarrow Z'(X_1) \longrightarrow B'(X_1) \longrightarrow 0$$

and

$$0 \longrightarrow B(X_2) \longrightarrow Z(X_2) \longrightarrow H(X_2) \longrightarrow 0$$

split. Then the homomorphism $\alpha\colon t(H(X_1), H(X_2)) \to Ht(X_1, X_2)$ is a monomorphism and the image of α is a direct summand of $Ht(X_1, X_2)$.

And similarly for left exact functors (without proof):

3.26 Proposition. Let $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be a left exact functor, covariant in the first and contravariant in the second variable, and let X_1 and X_2 be as in the previous proposition. Assume that the exact sequences

$$0 \longrightarrow B(X_1) \longrightarrow Z(X_1) \longrightarrow H(X_1) \longrightarrow 0$$

and

$$0 \longrightarrow H(X_2) \longrightarrow Z'(X_2) \longrightarrow B'(X_2) \longrightarrow 0$$

split. Then the homomorphism $\alpha'\colon Ht(X_1, X_2) \to t(H(X_1), H(X_2))$ is an epimorphism and the image of α' is a direct summand of $Ht(X_1, X_2)$. □

Proof of proposition 3.25. [CE56, p. 66] By composing the splitting homomorphisms $Z'(X_1) \to H(X_1)$ and $H(X_2) \to Z(X_2)$ with the canonical homomorphisms $X_1 \to Z'(X_1)$ and $Z(X_2) \to X_2$ one obtains the homomorphisms $\beta\colon X_1 \to H(X_1)$ and $\gamma\colon H(X_2) \to X_2$. The induced homomorphisms $H(X_1) \to H(X_1)$ and $H(X_2) \to H(X_2)$ are the identities. Hence one gets by the naturality of α the commutative diagram

$$\begin{array}{ccc} t(H(X_1), H(X_2)) & \xrightarrow{\text{id}} & t(H(X_1), H(X_2)) \\ \Big\downarrow \alpha & & \Big\downarrow \text{id} \\ Ht(X_1, X_2) & \xrightarrow[\xi]{} & t(H(X_1), H(X_2)) \end{array}$$

where the horizontal homomorphisms are induced by the homomorphism $t(\beta, \gamma)$. Hence $\xi\alpha = \text{id}$ which implies the claim. □

3.27 Corollary. In the case of theorem 3.19 the conditions of proposition 3.25 imply that the exact sequence 3.20

$$0 \longrightarrow t(H(X_1), H(X_2)) \xrightarrow{\alpha} Ht(X_1, X_2) \xrightarrow{\beta} t_1(H(X_1), H(X_2)) \longrightarrow 0$$

is split, that is

$$Ht(X_1, X_2) \cong t(H(X_1), H(X_2)) \oplus (t_1(H(X_1), H(X_2)))^+ \tag{3.28}$$

where $(t_1(H(X_1), H(X_2)))^+$ denotes the chain complex derived from the chain complex $t_1(H(X_1), H(X_2))$ by shifting the indexes $k \mapsto k - 1$. □

Dually, a similar result holds for theorem 3.21.

4 Application to $\otimes$ and Hom

There are two functors which tend to appear intrusively in algebra [Eil60]: the tensor product

$$\otimes\colon \mathbf{Mod}\text{–}R \times R\text{–}\mathbf{Mod} \to \mathbf{AG}$$

(which can also be seen as $\otimes\colon R\text{–}\mathbf{Mod} \times R\text{–}\mathbf{Mod} \to R\text{–}\mathbf{Mod}$ if the ring R is commutative) and the homomorphism functor

$$\mathrm{Hom}\colon R\text{–}\mathbf{Mod} \times R\text{–}\mathbf{Mod} \to \mathbf{AG}$$

For this section it is noteworthy to observe that given a ring R, the categories R–**Mod** of left R-modules and **Mod**–R of right R-modules do have projectives, injectives and coproducts. The category of abelian groups **AG**, being the special case $R = \mathbb{Z}$, has the same properties. [CE56, pp. 6–10]

4.1 On the Constraints of the Künneth Formula

The assumptions of the Künneth Formula as stated in theorems 3.19 and 3.21 include constraints on the derived functors as well as on the homomorphisms α and α' respectively. The aim in this section is to find constraints on the derived functors which imply the constraints on the homomorphisms α and α' as needed by the Künneth Formula.

Consider an additive functor $t\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$, covariant in the first variable and contravariant in the second variable. Assume that $\mathcal{A}$ has coproducts.

4.1 Proposition. Let t be right exact and assume that $\mathcal{A}_1$ has projectives and $\mathcal{A}_2$ has injectives. Let X_1 be a chain complex in $\mathcal{A}_1$ and X_2 a chain complex in $\mathcal{A}_2$ such that

$$\left.\begin{aligned} t_1(B(X_1), B(X_2)) = 0 = t_1(B(X_1), H(X_2)) \\ t_1(Z(X_1), B(X_2)) = 0 = t_1(Z(X_1), H(X_2)) \end{aligned}\right\} \tag{4.2}$$

Then requirements 1 of the Künneth Formula for right exact functors (theorem 3.19) are satisfied.

If on the other hand

$$\left.\begin{aligned} t_1(B(X_1), B'(X_2)) = 0 = t_1(H(X_1), B'(X_2)) \\ t_1(B(X_1), Z'(X_2)) = 0 = t_1(H(X_1), Z'(X_2)) \end{aligned}\right\} \tag{4.3}$$

then requirements 2 of theorem 3.19 are satisfied.

Proof. Consider first half of the theorem. Recall that one needs to show that the homomorphisms

$$\alpha_1\colon t(B(X_1), H(X_2)) \to Ht(B(X_1), X_2)$$

$$\alpha_2\colon t(Z(X_1), H(X_2)) \to Ht(Z(X_1), X_2)$$

are in fact isomorphisms and that

$$t_1(B(X_1), X_2) = 0 = t_1(Z(X_1), H(X_2))$$

The complex $t_1(Z(X_1), H(X_2)) = 0$ by assumption. In order to see that $t_1(B(X_1), X_2) = 0$ one first applies the functor $t_1(B(X_1), \bullet\,)$ (which is half exact by corollary 2.29) to the short exact sequence $0 \to B(X_2) \to Z(X_2) \to H(X_2) \to 0$. This yields the exact sequence

$$t_1(B(X_1), B(X_2)) \longrightarrow t_1(B(X_1), Z(X_2)) \longrightarrow t_1(B(X_1), H(X_2))$$

Since by assumption both $t_1(B(X_1), B(X_2)) = 0$ and $t_1(B(X_1), H(X_2)) = 0$ also $t_1(B(X_1), Z(X_2))$ is necessarily trivial. Using this result one concludes in a similar fashion from applying again the functor $t_1(B(X_1), \bullet\,)$, but this time to the short exact sequence $0 \to Z(X_2) \to X_2 \to B(X_2) \to 0$, that $t_1(B(X_1), X_2) = 0$.

In order to show that the homomorphisms α_1 and α_2 are indeed isomorphisms one considers for a while the functor $t(B(X_1), \bullet\,)$ and the functor $t_1(B(X_1), \bullet\,)$ as functors of the second variable only.

The complex $t_1(B(X_1), B(X_2))$ is trivial by assumption and from applying $t_1(B(X_1), \bullet\,)$ to the short exact sequence $0 \to B(X_2) \to Z(X_2) \to H(X_2) \to 0$ one concludes that $t_1(B(X_1), Z(X_2))$ is trivial, too. Moreover the homomorphisms

$$t(B(X_1), B(X_2)) \to Ht(B(X_1), B(X_2))$$

and

$$t(B(X_1), Z(X_2)) \to Ht(B(X_1), Z(X_2))$$

as in proposition 3.15 are the identities (since $B(X_2)$ and $Z(X_2)$ have trivial differential operators) and hence they are isomorphisms.

Altogether the requirements of theorem 3.19 (conditions 1) are satisfied.

Similarly for the second part of the claim. □

The dual proposition for left exact functors is

4.4 Proposition. Let t be left exact and assume that $\mathcal{A}_1$ has injectives and $\mathcal{A}_2$ has projectives. Let X_1 be a chain complex in $\mathcal{A}_1$ and X_2 a chain complex in $\mathcal{A}_2$ such that

$$\left.\begin{aligned} t'_1(B'(X_1), B'(X_2)) = 0 = t'_1(B'(X_1), H(X_2)) \\ t'_1(Z'(X_1), B'(X_2)) = 0 = t'_1(Z'(X_1), H(X_2)) \end{aligned}\right\} \tag{4.5}$$

Then requirements 1 of the Künneth Formula for left exact functors (theorem 3.21) are satisfied.

On the other hand, if the conditions

$$\left.\begin{aligned} t'_1(B'(X_1), B(X_2)) = 0 = t'_1(H(X_1), B(X_2)) \\ t'_1(B'(X_1), Z(X_2)) = 0 = t'_1(H(X_1), Z(X_2)) \end{aligned}\right\} \tag{4.6}$$

are satisfied then requirements 2 of theorem 3.21 are satisfied. □

If t is right exact and t_1 is left exact (or dually, if t is left exact and t_1' is right exact) one can simplify the previous constraints.

Consider first a right exact functor t as above and assume that condition 4.2 are satisfied.

Applying the functor $t_1(\bullet, B(X_2))$ to the short exact sequence $0 \to Z(X_1) \to X_1 \to B(X_1) \to 0$ yields the exact sequence

$$t_1(Z(X_1), B(X_2)) \longrightarrow t_1(X_1, B(X_2)) \longrightarrow t_1(B(X_2), B(X_2))$$

Then conditions 4.2 imply that $t_1(X_1, B(X_2)) = 0$.

Similarly applying $t_1(\bullet, H(X_2))$ to the exact sequence $0 \to Z(X_1) \to X_1 \to B(X_1) \to 0$ yields $t_1(X_1, H(X_2)) = 0$.

On the other hand, assume that

$$t_1(X_1, B(X_2)) = 0 = t_1(X_1, H(X_2)) \tag{4.7}$$

If the functor t_1 is assumed to be left exact, then applying $t_1(\bullet, B(X_2))$ and $t_1(\bullet, H(X_2))$ to the short exact sequence $0 \to Z(X_1) \to X_1 \to B(X_1) \to 0$ yields the exact sequences

$$0 \longrightarrow t_1(Z(X_1), B(X_2)) \longrightarrow t_1(X_1, B(X_2)) \longrightarrow t_1(H(X_1), B(X_2))$$

$$0 \longrightarrow t_1(Z(X_1), H(X_2)) \longrightarrow t_1(X_1, H(X_2)) \longrightarrow t_1(H(X_1), H(X_2))$$

and therefore

$$t_1(Z(X_1), B(X_2)) = 0 = t_1(Z(X_1), H(X_2))$$

With this one concludes from the short exact sequence $0 \to B(X_1) \to Z(X_1) \to H(X_1) \to 0$ in a similar way that

$$t_1(B(X_1), B(X_2)) = 0 = t_1(B(X_1), H(X_2))$$

and hence from conditions 4.7 follow conditions 4.2 due to the left exactness of t_1.

A similar conclusion shows that

$$t_1(B'(X_1), X_2) = 0 = t_1(H(X_1), X_2) \tag{4.8}$$

is equivalent with conditions 4.3, that is one has shown that

4.9 Proposition. Let t be an additive right exact functor as before and assume that t_1 is left exact. Let X_1 and X_2 be chain complexes. Then conditions 4.2 are equivalent with

$$t_1(X_1, B(X_2)) = 0 = t_1(X_1, H(X_2))$$

and conditions 4.3 are equivalent with

$$t_1(B'(X_1), X_2) = 0 = t_1(H(X_1), X_2)$$

□

Again, in a similar way one shows the dual

4.10 Proposition. Let t be an additive left exact functor as before and assume that t'_1 is right exact. Let X_1 and X_2 be chain complexes. Then conditions 4.5 are equivalent with

$$t'_1(X_1, B'(X_2)) = 0 = t'_1(X_1, H(X_2))$$

and conditions 4.6 are equivalent with

$$t'_1(B(X_1), X_2) = 0 = t'_1(H(X_1), X_2)$$

□

4.2 The Tensor Product ⊗

The functor $\otimes$: $\mathbf{Mod}\text{–}R \times R\text{–}\mathbf{Mod} \to \mathbf{AG}$ is defined in the usual way:

4.11 Definition. *Let R be a ring. Given a right R-module M_1 and a left R-module M_2, the* tensor product of M_1 and M_2 *(denoted by $M_1 \otimes M_2$ or $M_1 \otimes_R M_2$ if one needs to emphasize the underlying ring R) is the quotient group*

$$M_1 \otimes M_2 := F/E$$

where F is the free abelian group generated by the elements $x_1 \otimes x_2$ ($x_1 \in M_1$, $x_2 \in M_2$) and E is the subgroup of F generated by the elements of the form

$$\begin{aligned} &(x_1 + x'_1 \otimes x_2) - (x_1 \otimes x_2) - (x'_1 \otimes x_2) \\ &(x_1 \otimes x_2 + x'_2) - (x_1 \otimes x_2) - (x_1 \otimes x'_2) \\ &(x_1 r \otimes x_2) - (x_1 \otimes r x_2) \qquad\qquad (r \in R) \end{aligned}$$

If φ: $M_1 \to M'_1$ is a homomorphism of right R-modules and ψ: $M_2 \to M'_2$ is a homomorphism of left R-modules, then the homomorphism of abelian groups

$$\varphi \otimes \psi : M_1 \otimes M_2 \to M'_1 \otimes M'_2$$

is defined by

$$(\varphi \otimes \psi)(x_1 \otimes x_2) := \varphi(x_1) \otimes \psi(x_2)$$

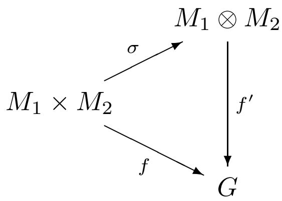
One verifies directly that the definition of the map $\varphi \otimes \psi$ indeed yields a well defined homomorphism and that the tensor product $\otimes$ is indeed a functor

$$\otimes\colon \mathbf{Mod}\text{–}R \times R\text{–}\mathbf{Mod} \to \mathbf{AG}$$

covariant in both variables. Clearly $\otimes$ is additive.

Moreover, the tensor product has the following characteristic property: Let M_1 be a right R-module and M_2 a left R-module. If now σ: $M_1 \times M_2 \to M_1 \otimes M_2$ is the map $(x_1, x_2) \mapsto x_1 \otimes x_2$ (which is bilinear by construction), then any bilinear map f: $M_1 \times M_2 \to G$ (where G is an abelian group) has a unique decomposition

$f = f'\sigma$. That is, there exists a homomorphism $f'\colon M_1 \otimes M_2 \to G$ such that the diagram

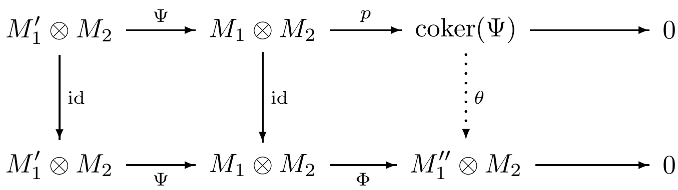


is commutative.

4.12 Proposition. The tensor product $\otimes\colon \mathbf{Mod}\text{–}R \times R\text{–}\mathbf{Mod} \to \mathbf{AG}$ is a right exact functor.

Proof. Given any short exact sequence $0 \to M_1' \to M_1 \to M_1'' \to 0$ in $\mathbf{Mod}$–R and $0 \to M_2' \to M_2 \to M_2'' \to 0$ in R–$\mathbf{Mod}$ one has to show that $M_1' \otimes M_2 \to M_1 \otimes M_2 \to M_1'' \otimes M_2 \to 0$ and $M_1 \otimes M_2' \to M_1 \otimes M_2 \to M_1 \otimes M_2'' \to 0$ are exact.

Consider the first case: Let φ and ψ be the homomorphisms $\varphi\colon M_1 \to M_1''$ and $\psi\colon M_1' \to M_1$, and denote the homomorphism $\varphi \otimes \mathrm{id}$ by Φ and the homomorphism $\psi \otimes \mathrm{id}$ by Ψ. It is enough to show that there exists an isomorphism $\theta\colon \operatorname{coker}(\Psi) \to M_1'' \otimes M_2$ making the diagram of abelian groups



(4.13)

with exact upper row commutative. This then forces the lower row to be exact, too.

Since $\Phi\Psi = (\varphi\psi) \otimes \mathrm{id} = 0$ one has $\operatorname{im}(\Psi) \subset \ker(\Phi)$ and hence there exists a homomorphism $\theta\colon \operatorname{coker}(\Psi) \to M_1'' \otimes M_2$, mapping $x + \operatorname{im}(\Psi) \mapsto \Phi(x)$.

To show that θ is an isomorphism a homomorphism θ' will be constructed, which then will be shown to be the inverse of θ. Therefore consider first the bilinear map

$$\eta\colon M_1'' \times M_2 \to \operatorname{coker}(\Psi)$$

defined as follows: Since φ is an epimorphism one can choose for any $x'' \in M_1''$ a $x \in M_1$ such that $\varphi(x) = x''$. Finally define $\eta(x'', y) := p(x \otimes y)$. This definition is independent of the choice of x and indeed defines a bilinear map. Hence there exists a unique factorization $\eta = \theta'\sigma$. Now clearly $\theta\theta' = \mathrm{id}$ and $\theta'\theta = \mathrm{id}$ and hence θ is a isomorphism. Hence the lower row in diagram 4.13 is exact, too.

The exactness of the sequence $M_1 \otimes M_2' \to M_1 \otimes M_2 \to M_1 \otimes M_2'' \to 0$ is shown in an analogous way. □

The left derived functors of the tensor product are denoted by Tor_i (or Tor_i^R if one needs to emphasize the underlying ring; $i \in \mathbb{N}$) and are called the *torsion functors*. Since the tensor product is right exact, proposition 2.30 states the existence of a natural equivalence $\mathrm{Tor}_0 \approx \otimes$.

Now is everything prepared to apply the Künneth Formula together with the considerations of the previous section 4.1 to the tensor product.

4.14 Theorem (Künneth Formula for Homology). Let C_1 be a chain complex of right R-modules and C_2 a chain complex of left R-modules. Assume that

$$\left.\begin{aligned} \operatorname{Tor}_1(B(C_1), B(C_2)) = 0 = \operatorname{Tor}_1(B(C_1), H(C_2)) \\ \operatorname{Tor}_1(Z(C_1), B(C_2)) = 0 = \operatorname{Tor}_1(Z(C_1), H(C_2)) \end{aligned}\right\} \tag{4.15}$$

Then there exists an exact sequence

$$0 \longrightarrow H(C_1) \otimes H(C_2) \xrightarrow{\alpha} H(C_1 \otimes C_2) \xrightarrow{\beta} \operatorname{Tor}_1(H(C_1), H(C_2)) \longrightarrow 0 \tag{4.16}$$

where α has degree zero and β has degree -1.

Proof. This follows straight from proposition 4.1 and theorem 3.19. □

A right R-module M_2 can be considered as a single graded complex with only one non-trivial component in dimension zero. Then $B(M_2) = 0$ and $H(M_2) = M_2$ and if one uses instead of the chain complex C_2 the module M_2 the constraints 4.15 simplify to

$$\operatorname{Tor}_1(B(C_1), M_2) = 0 = \operatorname{Tor}_1(Z(C_1), M_2) \tag{4.17}$$

Hence one gets immediately the

4.18 Corollary (Universal Coefficient Theorem for Homology). Let C_1 is a chain complex of right R-modules and M_2 a left R-module. If the conditions 4.17 are satisfied there is an exact sequence

$$0 \longrightarrow H(C_1) \otimes M_2 \xrightarrow{\alpha} H(C_1 \otimes M_2) \xrightarrow{\beta} \operatorname{Tor}_1(H(C_1), M_2) \longrightarrow 0 \tag{4.19}$$

where α has degree zero and β has degree -1. □

Using definition 1.20 to write down the sequence 4.16 of the Künneth Formula explicitly yields the short exact sequences

$$\begin{aligned} 0 \longrightarrow \sum_{i_1+i_2=i} H_{i_1}(C_1) \otimes H_{i_2}(C_2) \xrightarrow{\alpha_i} H_i(C_1 \otimes C_2) \\ \xrightarrow{\beta_i} \sum_{i_1+i_2=i} \operatorname{Tor}_1(H_{i_1-1}(C_1), H_{i_2}(C_2)) \longrightarrow 0 \qquad (i \in \mathbb{Z}) \end{aligned}$$

If the chain complex C_2 has only one non-trivial module M_2 at dimension zero these sequences simplify to

$$0 \longrightarrow H_i(C_1) \otimes M_2 \xrightarrow{\alpha_i} H_i(C_1 \otimes M_2) \xrightarrow{\beta_i} \operatorname{Tor}_1(H_{i-1}(C_1), M_2) \longrightarrow 0 \qquad (i \in \mathbb{Z})$$

4.20 Definition. *A ring R is said to be* left semi-hereditary *if every finitely generated submodule of a projective left R-module is itself projective. Similarly,* *a ring R is said to be* right semi-hereditary *if every finitely generated submodule of a projective right R-module is itself projective.*

A ring R is said to be left (right) hereditary *if every submodule of a projective left (respectively right) R-module is a projective module.*

Now if the ring R is left (or right) semi-hereditary this has implications on the functor Tor_1: the functor becomes left exact (see appendix C). Then proposition 4.9 states that conditions 4.15 in theorem 4.14 are equivalent with

$$\mathrm{Tor}_1(C_1, B(C_2)) = 0 = \mathrm{Tor}_1(C_1, H(C_1))$$

Due to lemma 2.25 these conditions are satisfied if C_1 is a projective chain complex. If R is assumed to be left hereditary then these conditions are already satisfied if C_1 or C_2 is a projective chain complex.

If the ring R is left and right hereditary and the complexes C_1 and C_2 are projective, then complexes $B(C_1)$ and $B(C_2)$ are projective, too, and therefore the short exact sequences

$$0 \to B(C_1) \to Z(C_1) \to H(C_1) \to 0$$

$$0 \to B(C_2) \to Z(C_2) \to H(C_2) \to 0$$

are split and one can apply corollary 3.27 to get the

4.21 PROPOSITION. If R is a left and right hereditary ring and C_1 and C_2 are projective chain complexes, then the exact sequence 4.16

$$0 \longrightarrow H(C_1) \otimes H(C_2) \longrightarrow H(C_1 \otimes C_2) \longrightarrow \mathrm{Tor}_1(H(C_1), H(C_2)) \longrightarrow 0$$

is split. □

And similarly applying the above considerations to the Universal Coefficient Theorem (corollary 4.18) yields

4.22 PROPOSITION. Let C_1 be a chain complex and M_2 a left R-module. If R is left or right semi-hereditary, conditions 4.17 are equivalent to $\mathrm{Tor}_1(C_1, M_2) = 0$. If R is right hereditary and C_1 is projective then the exact sequences

$$0 \longrightarrow H_i(C_1) \otimes M_2 \xrightarrow{\alpha_i} H_i(C_1 \otimes M_2) \xrightarrow{\beta_i} \mathrm{Tor}_1(H_{i-1}(C_1), M_2) \longrightarrow 0 \qquad (i \in \mathbb{Z})$$

are split. □

4.3 The Homomorphism Functor Hom

In the usual way one extends the mapping $\mathrm{Hom}\colon\ R\text{–}\mathbf{Mod} \times R\text{–}\mathbf{Mod} \to \mathbf{AG}$ to a functor:

4.23 DEFINITION. *Let M_1, M_1', M_2 and M_2' be left R-modules. Then define for any homomorphisms $\varphi\colon M_1 \to M_1'$ and $\psi\colon M_2' \to M_2$ the homomorphism*

$$\mathrm{Hom}(\varphi, \psi)\colon\ \mathrm{Hom}(M_1, M_2) \to \mathrm{Hom}(M_1', M_2')$$

by

$$\mathrm{Hom}(\varphi, \psi)(\alpha) := \psi \circ \alpha \circ \varphi \qquad (\alpha \in \mathrm{Hom}(M_1, M_2))$$

Observe that this defines indeed a functor $\mathrm{Hom}\colon R\text{–}\mathbf{Mod} \times R\text{–}\mathbf{Mod} \to \mathbf{AG}$ (or Hom_R if one needs to emphasize the underlying ring R). This functor is additive, contravariant in the first and covariant in the second variable.

4.24 Proposition. The functor Hom is left exact.

Proof. First, given a short exact sequence $0 \to M_1' \xrightarrow{\varphi} M_1 \xrightarrow{\psi} M_1'' \to 0$ of left R-modules and a left R-module M_2 one needs to show that the sequence

$$0 \longrightarrow \mathrm{Hom}(M_1'', M_2) \xrightarrow{\Psi} \mathrm{Hom}(M_1, M_2) \xrightarrow{\Phi} \mathrm{Hom}(M_1', M_2)$$

is exact.

The approach is similar to the proof of proposition 4.12: it is enough to show that there exists an isomorphism $\theta\colon \mathrm{Hom}(M_1'', M_2) \to \ker(\Phi)$ making the diagram of abelian groups

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(M_1'', M_2) & \xrightarrow{\Psi} & \mathrm{Hom}(M_1, M_2) & \xrightarrow{\Phi} & \mathrm{Hom}(M_1', M_2) \\
 & & \vdots\,\theta & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\
0 & \longrightarrow & \ker(\Phi) & \xrightarrow[i]{} & \mathrm{Hom}(M_1, M_2) & \xrightarrow[\Phi]{} & \mathrm{Hom}(M_1', M_2)
\end{array}$$

with exact lower row commutative. This forces then the upper row to be exact, too.

Since $\Phi\Psi = 0$ the homomorphism Ψ induces a homomorphism

$$\theta\colon \mathrm{Hom}(M_1'', M_2) \to \ker(\Phi)$$

To see that θ is indeed an isomorphism one again construct explicitly its inverse:

Therefore define a homomorphism

$$\theta'\colon \ker(\Phi) \to \mathrm{Hom}(M_1'', M_2)$$

as follows: Let $f \in \mathrm{Hom}(M_1, M_2)$ such that $\Phi(f) = \mathrm{id} \circ f \circ \varphi = 0$. Now for $x'' \in M_1''$ define $\theta'(f)(x'') := f(x)$ where x is any $x \in M_1$ such that $\psi(x) = x''$. Observe that this indeed defines a homomorphism $\theta'\colon \ker(\Phi) \to \mathrm{Hom}(M_1'', M_2)$ and that $\theta\theta' = \mathrm{id}$ and $\theta'\theta = \mathrm{id}$ and therefore θ is a isomorphism.

Similarly the left exactness is proven for the second variable. □

The right derived functors of Hom are *extension functors* Ext^i (or Ext^i_R if one needs to emphasis the underlying ring R; $i \in \mathbb{N}$). Since Hom is left exact there exists by proposition 2.33 a natural equivalence $\mathrm{Ext}^0 \approx \mathrm{Hom}$.

Again everything is now prepared to apply the Künneth Formula together with the considerations of section 4.1 to the homomorphism functor.

4.25 Theorem (Künneth Formula for Cohomology). Let C_1 and C_2 be chain complexes of left R-modules. Assume that

$$\left.\begin{aligned} \operatorname{Ext}^1(B(C_1), B'(C_2)) = 0 = \operatorname{Ext}^1(B(C_1), H(C_2)) \\ \operatorname{Ext}^1(Z(C_1), B'(C_2)) = 0 = \operatorname{Ext}^1(Z(C_1), H(C_2)) \end{aligned}\right\} \tag{4.26}$$

Then there exists an exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Ext}^1(H(C_1), H(C_2)) \xrightarrow{\beta'} H \operatorname{Hom}(C_1, C_2) \\ \xrightarrow{\alpha'} \operatorname{Hom}(H(C_1), H(C_2)) \longrightarrow 0 \end{aligned} \tag{4.27}$$

with β' having degree -1 and α' having degree zero.

Proof. This follows directly from proposition 4.4 and theorem 3.21. □

Again, a left R-module M_2 is considered as a single graded complex with only one non-trivial component in dimension zero. Then if one uses the module M_2 instead of the chain complex C_2 the constraints 4.26 simplify to

$$\operatorname{Ext}^1(B(C_1), M_2) = 0 = \operatorname{Ext}^1(Z(C_1), M_2) \tag{4.28}$$

As a direct consequence of theorem 4.25 one gets the

4.29 Corollary (Universal Coefficient Theorem for Cohomology). Assume that C_1 is a chain complex of left R-modules and M_2 a left R-module. If the conditions 4.28 are satisfied there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}^1(H(C_1), M_2) \xrightarrow{\beta'} H \operatorname{Hom}(C_1, M_2) \xrightarrow{\alpha'} \operatorname{Hom}(H(C_1), M_2) \longrightarrow 0 \tag{4.30}$$

with β' having degree -1 and α' having degree zero. □

Using definition 1.20 to write down explicitly the sequence 4.27 of the Künneth Formula for Cohomology yields the short exact sequences

$$\begin{aligned} 0 \longrightarrow \sum_{i_1+i_2=i+1} \operatorname{Ext}^1(H_{-i_1}(C_1), H_{i_2}(C_2)) \xrightarrow{\beta'_i} H_i \operatorname{Hom}(C_1, C_2) \\ \xrightarrow{\alpha'_i} \sum_{i_1+i_2=i} \operatorname{Hom}(H_{-i_1}(C_1), H_{i_2}(C_2)) \longrightarrow 0 \qquad (i \in \mathbb{Z}) \end{aligned}$$

Since

$$\begin{aligned} \sum_{i_1+i_2=i+1} \operatorname{Ext}^1(H_{-i_1}(C_1), H_{i_2}(C_2)) &= \sum_{(i_1-1)+i_2=i} \operatorname{Ext}^1(H_{-(i_1-1)-1}(C_1), H_{i_2}(C_2)) \\ &= \sum_{i_1+i_2=i} \operatorname{Ext}^1(H_{-i_1-1}(C_1), H_{i_2}(C_2)) \end{aligned}$$

these sequences are equal to

$$0 \longrightarrow \sum_{i_1+i_2=i} \operatorname{Ext}^1(H_{-i_1-1}(C_1), H_{i_2}(C_2)) \xrightarrow{\beta_i'} H_i \operatorname{Hom}(C_1, C_2)$$

$$\xrightarrow{\alpha_i'} \sum_{i_1+i_2=i} \operatorname{Hom}(H_{-i_1}(C_1), H_{i_2}(C_2)) \longrightarrow 0 \qquad (i \in \mathbb{Z})$$

If the chain complex C_2 is replaced by a left module M_2 these sequences simplify to

$$0 \longrightarrow \operatorname{Ext}^1(H_{-i-1}(C_1), M_2) \xrightarrow{\beta_i'} H_i \operatorname{Hom}(C_1, M_2)$$

$$\xrightarrow{\alpha_i'} \operatorname{Hom}(H_{-i}(C_1), M_2) \longrightarrow 0 \qquad (i \in \mathbb{Z})$$

If R is a left hereditary ring, then Ext^1 becomes a right exact functor (proposition C.4). Then proposition 4.10 states that the conditions 4.26 in theorem 4.25 are equivalent with

$$\operatorname{Ext}^1(C_1, B(C_2)) = 0 = \operatorname{Ext}^1(C_1, H(C_1))$$

In particular, these constraints are satisfied if C_1 is projective (lemma 2.25).

Analogous to the proof of proposition 4.21 one shows

4.31 Proposition. If R is a left hereditary ring and C_1 is a projective chain complex and C_2 an injective chain complex of left R-modules, then the exact sequence 4.27

$$0 \longrightarrow \operatorname{Ext}^1(H(C_1), H(C_2)) \xrightarrow{\beta'} H \operatorname{Hom}(C_1, C_2)$$

$$\xrightarrow{\alpha'} \operatorname{Hom}(H(C_1), H(C_2)) \longrightarrow 0$$

splits. □

A Members of Objects in Abelian Categories

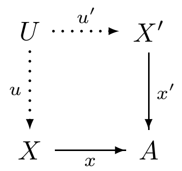
Many diagram lemmas are proved in the category of abelian groups **AG** by diagram chasing.

In arbitrary abelian categories this method cannot be used directly due to the lack of elements in an object. But the concept of elements can partially be replaced by the concept of members as described for example in [ML71, pp. 200ff.].

A.1 Definition. *Let A be an object of an abelian category $\mathcal{A}$. A* member *x of A is any homomorphism $x\colon X \to A$, in symbols $x \in_m A$.*

Two members x and $x' \in_m A$ are said to be equivalent*, if there exists epimorphisms u and u' such that $xu = x'u'$, in symbols $x \equiv x'$.*

The situation of two equivalent members can be illustrated as follows: if $x\colon X \to A$ and $x'\colon X' \to A$ are members of A, then these members are equivalent if there exist epimorphisms which make the diagram



commute.

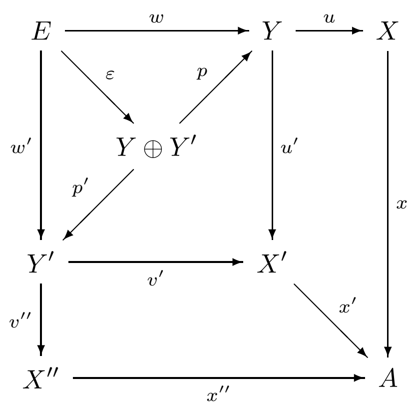
A.2 Lemma. The relation "$\equiv$" in definition A.1 is a equivalence relation.

A.3 Remark. With this result one could be tempted to start forming equivalence classes on the members of an object, however this approach would have its difficulties since the equivalence classes would in general turn out to be proper classes which could not be members of another class.

Proof of lemma A.2. Clearly "$\equiv$" is reflexive and symmetric. Only the transitivity remains to be shown.

Let $x\colon X \to A$, $x'\colon X' \to A$ and $x''\colon X'' \to A$ be members of A, and $u\colon Y \to X$, $u'\colon Y \to X'$, $v'\colon Y' \to X'$ and $v''\colon Y' \to X''$ epimorphisms such that $xu = x'u'$ and $x'v' = x''v''$.

Consider the commutative diagram



(A.4)

where p and p' are the projections of the direct sum $Y \oplus Y'$ and $\varepsilon\colon E \to Y \oplus Y'$ is the equalizer of the homomorphisms $v'p'$ and vp. The equalizer exists since abelian categories have equalizers (abelian categories are normal and have kernels and finite products [Mit65, p. 33], normal categories with kernels have finite intersections [Mit65, p. 16] and finally categories with finite intersections and finite products have equalizers [Mit65, p. 27]).

Then the rectangle

$$\begin{array}{ccc} E & \xrightarrow{w} & Y \\ {\scriptstyle w'}\downarrow & & \downarrow{\scriptstyle u'} \\ Y' & \xrightarrow[v']{} & X' \end{array}$$

is a pullback [Mit65, p. 27] and since u' and v' are epimorphisms, also w and w' are epimorphisms [Mit65, p. 34].

Diagram A.4 is commutative and hence $x''(v''w') = x(uw)$ with $v''w'$ and uw being epimorphisms, that is $x \equiv x''$. □

The next theorem (see also [ML71, p. 200]) describes the elementary rules for diagram chasing in arbitrary abelian categories:

A.5 Theorem. Let $\mathcal{A}$ be an abelian category and $f\colon A \to B$ and $g\colon B \to C$ homomorphisms of $\mathcal{A}$. Then

1. f is a monomorphism if and only if for all $x \in_m A$ the condition $fx \equiv 0$ implies that $x \equiv 0$,
2. f is a monomorphism if and only if for all $x, x' \in_m A$ the condition $fx \equiv fx'$ implies that $x \equiv x'$,
3. f is an epimorphism if and only if for each $y \in_m B$ there exists a $x \in_m A$ such that $fx \equiv y$,
4. f is the zero homomorphism if and only if for all $x \in_m A$ holds $fx \equiv 0$,
5. the sequence
$$A \xrightarrow{f} B \xrightarrow{g} C$$
is exact if and only if $gf = 0$ and for every $y \in_m B$ with $gy \equiv 0$ there exists a $x \in_m A$ such that $fx \equiv y$, and
6. for fixed x and $x' \in_m B$ such that $gx \equiv gx'$ there exists a $z \in_m B$ such that $gz \equiv 0$. Moreover, if one has $hx \equiv 0$ for a homomorphism $h\colon B \to D$, then $hx' \equiv hz$, and similarly, if one has $h'x' \equiv 0$ for a homomorphism $h'\colon B \to A$ then $h'x \equiv -h'z$.

Proof. See [ML71, p. 201]. □

B The Snake Lemma

As an application of the method of members, I present a proof of the Snake Lemma in a general abelian category. This result is used in this thesis in section 2.3 to show the existence of certain long exact homology sequences (see theorem 2.26).

Assume the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1' & \xrightarrow{\psi_1} & A_1 & \xrightarrow{\varphi_1} & A_1'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & A_2' & \xrightarrow[\psi_2]{} & A_2 & \xrightarrow[\varphi_2]{} & A_2'' & \longrightarrow & 0
\end{array}$$

with exact rows. Then the homomorphisms ψ_k and φ_k induce homomorphisms $\psi'\colon \ker(f') \to \ker(f)$ and $\varphi'\colon \ker(f) \to \ker(f'')$ respectively. Similarly homomorphisms $\bar{\psi}\colon \operatorname{coker}(f') \to \operatorname{coker}(f)$ and $\bar{\varphi}\colon \operatorname{coker}(f) \to \operatorname{coker}(f'')$ are induced by ψ_k and φ_k.

With this one can extend the previous diagram to the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(f') & \xrightarrow{\psi_1'} & \ker(f) & \xrightarrow{\varphi_1'} & \ker(f'') & & \\
 & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i''} & & \\
0 & \longrightarrow & A_1' & \xrightarrow{\psi_1} & A_1 & \xrightarrow{\varphi_1} & A_1'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & A_2' & \xrightarrow[\psi_2]{} & A_2 & \xrightarrow[\varphi_2]{} & A_2'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p''} & & \\
 & & \operatorname{coker}(f') & \xrightarrow{\bar{\psi}_1} & \operatorname{coker}(f) & \xrightarrow{\bar{\varphi}_1} & \operatorname{coker}(f'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

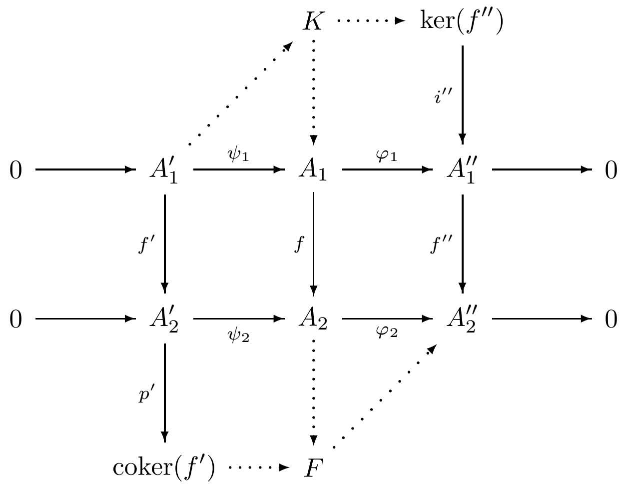
with exact columns. By chasing members it is straight forward to see that the added rows are exact.

B.1 Lemma (Snake Lemma). In the situation described above there exists a homomorphism $\delta\colon \ker(f'') \to \operatorname{coker}(f')$ such that the sequence

$$0 \to \ker(f') \to \ker(f) \to \ker(f'') \xrightarrow{\delta} \operatorname{coker}(f') \to \operatorname{coker}(f) \to \operatorname{coker}(f'') \to 0$$

is exact.

Proof. [ML71, p. 201] With the given short exact sequences and homomorphisms one first construct the slightly different diagram



Here K is the pullback of the homomorphisms i'' and φ_1. Since the homomorphism φ_1 is an epimorphism, the homomorphism $K \to \ker(f'')$ is also an epimorphism.

Similarly, F is the pushout of p' and ψ_2, and since ψ_2 is a monomorphism so is $\operatorname{coker}(f') \to F$ a monomorphism, too.

Denote the composition $K \to A_1 \to A_2 \to F$ by δ_0 and observe that $\delta_0 \circ (A_1' \to K) = (\operatorname{coker}(f') \to F) \circ p' \circ f' = 0$ and similarly $(F \to A_2'') \circ \delta_0 = 0$. As a consequence of $\operatorname{coker}(f') \to F$ being the kernel of $F \to A_2''$ and $K \to \ker(f'')$ being the cokernel of $A_1' \to F$ the homomorphism δ_0 factors uniquely as

$$\delta_0 = (\operatorname{coker}(f') \to F) \circ \delta \circ (K \to \ker(f''))$$

The middle factor $\delta\colon \ker(f'') \to \operatorname{coker}(f')$ is the required *connecting homomorphism.*

Now the effect of the connecting homomorphism δ can be described as follows (compare with figure 3):

Let x'' be a member of $\ker(f'') \subset A_1''$. Then there exists a $x \in_m A_1$ such that $\varphi_1 \circ x \equiv i'' \circ x''$. Clearly $\varphi \circ f \circ x \equiv 0$ and therefore one can find a $x' \in_m A_2'$ such that $\psi_2 \circ x' \equiv f \circ x$. Now $\delta \circ x'' \equiv p' \circ x'$.

Proving the exactness of exact sequence

$$0 \to \ker(f') \to \ker(f) \to \ker(f'') \xrightarrow{\delta} \operatorname{coker}(f') \to \operatorname{coker}(f) \to \operatorname{coker}(f'') \to 0$$

is now easily done by chasing members. [ML71, p. 201] □

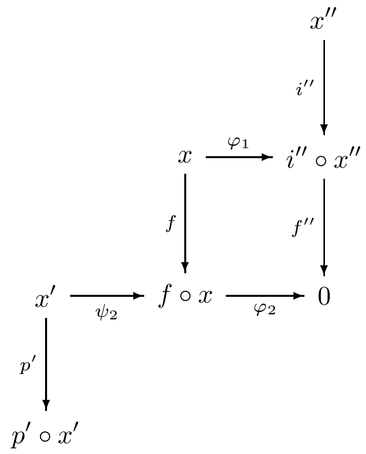


Figure 3: The effect of the connecting homomorphism on a member of $\ker(f'')$.

C Exactness of Tor_1 and Ext^1

In general the first torsion functor Tor_1 is not left exact nor is the first extension functor Ext^1 in general right exact. But under certain constraints for the ring R these functors become left (respectively right) exact.

This appendix tries to frame the reason why this is, but the attempt is rough and mostly without exact proof. It is mainly based on [CE56, pp. 97–100] and [CE56, pp. 106–112].

The key point in both cases is to demand a certain kind of hereditariness of the ring R. I use the definition 4.20 of semi-hereditary and hereditary ring, which is equivalent to the definition in [CE56, pp. 12–15].

C.1 Left or Right Semi-Hereditary Rings and Tor_1

The tensor product $\otimes$ is a functor of type $L\Sigma^*$ [CE56, p. 99], that is $\otimes$ commutes with colimits.

Now one can prove the

C.1 Proposition. If A is the colimit of a colimiting system A_α then there exists a colimiting system X_α of projective resolutions of A_α such that $X = \operatorname{colim} X_\alpha$ is a projective resolution of A. [CE56, p. 100] □

As an immediate consequence of this one gets the result that if t is a covariant functor (in any number of variables) of type $L\Sigma^*$, then the same is true for the left derived functors t_i. [CE56, p. 100]

In particular, the torsion functors Tor_i are of type $L\Sigma^*$, that is the torsion functors commute with colimits.

Now if R is a left or right semi-hereditary ring then every finitely generated submodule of a projective module is itself projective.

If R is left semi-hereditary and given a right R-module M_1 and a left R-module M_2 there exists an exact sequence $0 \to K \to P \to M_2 \to 0$ with P being projective ($K = \ker(P \to M_2)$). Applying corollary 2.31 yields $\mathrm{Tor}_i(M_1, K) \cong \mathrm{Tor}_{i-1}(M_1, K)$ $(i > 1)$. Any finitely generated submodule $K' \subset K \subset P$ is by assumption projective (since R is left semi-hereditary) and hence $\mathrm{Tor}_{i-1}(M_1, K')$ is trivial. Since the torsion functors are of type $L\Sigma^*$ also $\mathrm{Tor}_{i-1}(M_1, K) = 0$ for $i > 1$. Similar considerations hold in the case of R being right semi-hereditary. [CE56, p. 112]

Hence one gets

C.2 Proposition. If the ring R is (left or right) semi-hereditary then $\mathrm{Tor}_i = 0$ for $i > 1$. In particular Tor_1 is left exact. □

C.2 Left Hereditary Rings and Ext^1

Although the functor Hom is of type $R\Pi^*$ [CE56, p.99], that is the functor Hom commutes with limits, there exists no pendant for injective resolutions and limits to proposition C.1. [CE56, p. 100] Hence one cannot assume in the same way that the derived functors Ext^i inherit the property $R\Pi^*$ from the functor Hom as has been done in case of the derived functors of the tensor product $\otimes$ and the property $L\Sigma^*$.

C.3 Definition. *A left R-module M is said to have* projective dimension *less than n if there exists a projective resolution X of M such that $X_i = 0$ if $i > n$. The least such integer n is said to be the* projective dimension *of M, in symbols* $\mathrm{P-dim}\, M = n$.

Now a left R-module M has projective dimension less than n if and only if for any exact sequence $0 \to X_n \to X_{n-1} \to \ldots \to X_0 \to M$ with X_i projective $(0 \leq i < n)$ it follows that necessarily X_n is projective, too. [CE56, p. 110]

The least integer n such that for any left R-module M the projective dimension less than n is said to be the *left global dimension* of R. [CE56, p. 111]

If the ring R has left global dimension of n, then clearly $\mathrm{Ext}^i = 0$ for $i > n$ and in particular Ext^i is right exact for $i \geq n$.

From the above one obtains straight

C.4 Proposition. A ring R is left hereditary if and only if its left global dimension less than 1. [CE56, p. 112]

In particular the functor Ext^1 is right exact if R is left hereditary. □

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