P.A. SMITH THEORY FOR p-ADIC TRANSFORMATION GROUPS

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Abstract. This paper describes in short the construction of a modified version of Smith Theory developed by Chung-Tao Yang in his article on p-adic transformation groups in the early 60's [Yan60]. Yang uses this modified Smith Theory to derive some interesting results about the homology dimension of the orbit spaces of a homology n-manifold under the effective action of the group of p-adic integers, p-prime. Yang's result has recently been successfully used to give affirmative answers to the Hilbert-Smith Conjecture in special cases. It is believed that Yang's result might even be usefull to prove this conjecture in its full generality.

The content of this text is my licentiate's thesis which I have completed in 2007 under the supervision of Prof. Sören Illman at the university of Helsinki.

1. The Hilbert–Smith Conjecture and Yang's Contribution

The Hilbert–Smith conjecture – which is a generalized version of Hilbert's Fifth Problem, see [Hil00] – states that among the locally compact groups only Lie groups can act effectively on a finite dimensional manifold. More precisely:

Conjecture (Hilbert–Smith)**.** *Let M be a connected manifold. Assume that G is a locally compact group acting effectively on M. Then G can be given the structure of a Lie-group.*

Though the original problem is solved affirmative the Hilbert–Smith conjecture is only proven in parts and still open in its full generality. If one would be able to prove this conjecture then the differentiability assumption in conjunction with continuous group actions would turn out to be redundant.

Newmann [New31] and Smith [Smi41] have shown that a non-Lie group G acting effectively on a finite dimensional manifold necessarily contains a subgroup isomorphic to the group of p-adic integers $\mathcal{Z}_p$ for some prime p. Thus the Hilbert–Smith conjecture can be proven if it is possible to show, that the p-adic integers cannot act effectively on any finite dimensional manifold.

In 1960 C.T. Yang proved in his article about p-adic transformation groups [Yan60] the following result.

Theorem. *Let M be a homology n-manifold. Assume that the group of p-adic integers $\mathcal{Z}_p$ is acting effectively on M. Then the homology dimension of the orbit space $M/\mathcal{Z}_p$ is precisely $n+2$.* □

Now Yang's paper yields a possible approach to prove the Hilbert–Smith conjecture: if it is possible to show that the canonical projection map $\pi\colon M \to M/\mathcal{Z}_p$ onto the orbit space does not raise the homology dimension by 2 then this leads to a contradiction with Yang's result and thus proves the conjecture.

Though Yang's result is already known for long, it was not before the late 90's that it was used in an effective way. Repovš and Ščepin showed in 1997 using the above reasoning that the Hilbert–Smith conjecture holds for Lipschitz maps [RŠ97] and in the same year Malešič affirmed the conjecture for Hölder actions [Mal97]. As

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a last example for the successful application of Yang's result may serve Martin article from the year 1999 [Mar99] where he proved the conjecture for quasiconformal actions.

2. The Classical Smith Theory

The the classical Smith Theory was developed by P.A. Smith in a series of articles [Smi38, Smi39, Smi41, Smi45] when studying the homological properties of periodic transformations of prime order p on Euclidean n-spaces or equivalently n-spheres.

For a fixed prime p Smith studied simplicial C_p-complexes and their homology groups with coefficients in $\mathbb{Z}_p$. Here C_p denotes the cyclic group of p elements, written multiplicatively, and $\mathbb{Z}_p$ denotes the field of the integers modulo p. The simplicial C_p-complex K is assumed to satisfy certain regularity conditions [Kaw91, p. 229] which ensure that the fixed point set K^{C_p} is a simplicial complex and which ensure that we can define a simplicial orbit space K/C_p which has in a natural way again the structure of a simplicial complex. Such C_p-complexes shall be called in the following *regular* C_p-complexes. The restriction to regular C_p-complexes is not a very strong requirement since one can see that the second barycentric subdivision of any C_p-complex is regular.

We fix a generator g of C_p and define two special element of the group ring $\mathbb{Z}_p[C_p]$ of C_p over $\mathbb{Z}_p$ by

$$\begin{aligned} \sigma &:= 1 + g + g^2 + \ldots + g^{p-1}, \\ \tau &:= 1 - g. \end{aligned}$$

Since $g^p = 1$ it follows that σ and τ are zero divisors in $\mathbb{Z}_p[C_p]$. The action of C_p on K induces an action of C_p on the chain complex $C(K)$ of ordered simplices of K. We get in a natural way a homomorphism from $\mathbb{Z}_p[C_p]$ to the endomorphism ring $\operatorname{End}(C(K;\mathbb{Z}_p))$ of the chain complex of ordered simplices of K with coefficients in $\mathbb{Z}_p$.

Now $\ker\sigma$, $\operatorname{im}\sigma$, $\ker\tau$ and $\operatorname{im}\tau$ are subcomplexes of $C(K;\mathbb{Z}_p)$ and it turns out that their homology groups is an essential part of Smith Theory. This leads to the following definition.

Definition. The *special homology groups* K with respect to σ and τ are the following four graded groups:

$$\begin{aligned} H^\sigma(K) &:= H(\ker\sigma) & \overline{H}^\sigma(K) &:= H(\operatorname{im}\sigma) \\ H^\tau(K) &:= H(\ker\tau) & \overline{H}^\tau(K) &:= H(\operatorname{im}\tau) \end{aligned}$$

In Smith Theory these groups are used to study the relationship of the homology groups of the simplicial complexes K, K^{C_p} and K/C_p with coefficients in $\mathbb{Z}_p$.

3. p-Adic Integers

In the following p shall denote a fixed prime and we use the abbreviation $[r]$ to denote p^r for some $r \in \mathbb{N}$.[1] Recall the standard definition of the group of p-adic integers which we will denote by $\mathcal{Z}_p$.[2]

[1] The set of natural numbers $\mathbb{N}$ shall contain the 0.

[2] In the literature the group of p-adic integers is commonly denoted by A_p.

Definition (p-Adic Integers)**.** Consider the set of natural numbers $\mathbb{N}$ together with the natural order "$\leq$" as a directed set. For $i, j \in \mathbb{N}$, $i \leq j$ let

$$h_i^j\colon \mathbb{Z}_{[j]} \to \mathbb{Z}_{[i]}$$

be the canonical projection. Then the limit group of the inverse system $\{\mathbb{Z}_{[i]}, h_i^j\}$ directed by $\mathbb{N}$ is called the group of *p-adic integers* $\mathcal{Z}_p$, that is

$$\mathcal{Z}_p := \varprojlim \mathbb{Z}_{[i]}.$$

We shall collect some facts about the group of p-adic integers and introduce some more notation. The group $\mathcal{Z}_p$ is compact[3] and totally disconnected. We denote by h_i the projections

$$h_i\colon \mathcal{Z}_p \to \mathbb{Z}_{[i]}, \qquad i \in \mathbb{N}$$

which are continuous epimorphisms of groups. Furthermore we denote by $\mathbf{Z}_i := \ker(h_i)$ the kernels of these homomorphism. These subgroups are both closed and open in $\mathcal{Z}_p$. Furthermore those subgroups form a descending sequence of subgroups

$$\mathcal{Z}_p = \mathbf{Z}_0 \supset \mathbf{Z}_1 \supset \mathbf{Z}_2 \supset \mathbf{Z}_3 \supset \ldots$$

such that $\mathbb{Z}_{[i]} \cong \mathcal{Z}_p/\mathbf{Z}_i$. This sequence forms an open neighboorhod system of the identity element of $\mathcal{Z}_p$. If H is a open subgroup of $\mathcal{Z}_p$ then $H = \mathbf{Z}_i$ for some $i \in \mathbb{N}$. Similarly, if H is a closed subgroup, then either $H = \mathbf{Z}_i$ for some $i \in \mathbb{N}$ or H is the trivial subgroup of $\mathcal{Z}_p$. And last but not least the open subgroups of $\mathcal{Z}_p$ are all topological isomorphic.

4. A Modified Version of Smith Theory

In order to extend the Smith Theory to p-adic actions on compact spaces we need to be able to consider periodic transformation of order $[r] = p^r$ for different values $r \in \mathbb{N}$. Furthermore we need to be able to deal with coefficient groups $\mathbb{Z}_{[r]}$ for different values $r \in \mathbb{N}$ at the same time. Yang solves the latter problem by using the reals modulo the integers $\mathcal{R} := \mathbb{R}/\mathbb{Z}$ as the coefficient group. Since we will nearly exclusively deal with $\mathcal{R}$ as the coefficient groups we shall omit them from the notation from now on.

Thus we consider now the following situation: Let K be a finite regular simplicial G-complex where G is a cyclic p-group, say $G = C_{[r]}$ for some $r \in \mathbb{N}$. We fix a generator g_0 of the group G.

In the group ring $\mathbb{Z}[G]$ of G over $\mathbb{Z}$ we define two special elements, namely

$$\begin{aligned} \sigma &:= 1 + g_0 + g_0^2 + \ldots + g_0^{[r]-1}, \\ \tau &:= 1 - g_0. \end{aligned}$$

It follows again that σ and τ are zero-divisors in $\mathbb{Z}[G]$, that is $\sigma\tau = 0$ and $\tau\sigma = 0$.

The action of G on K makes the chain complex $C(K)$ of ordered simplices of K and coefficients in $\mathbb{R}$ into a graded $\mathbb{Z}[G]$-module. The above defined elements define chain maps $C(K) \to C(K)$. Since σ and τ are zero-divisors there exists inclusions $\imath\colon \operatorname{im}\tau \hookrightarrow \ker\sigma$ and $\imath'\colon \operatorname{im}\sigma \hookrightarrow \ker\tau$. Further we denote by ω and ω' the inclusion of $\ker\sigma$ and $\ker\tau$ respectively into $C(K)$. We obtain the following

[3] Compactness shall always imply the Hausdorff condition in this text.

short exact sequences of chain complexes:

$$0 \longrightarrow \ker\sigma \xrightarrow{\omega} C(K) \xrightarrow{\sigma} \operatorname{im}\sigma \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow \ker\tau \xrightarrow{\omega'} C(K) \xrightarrow{\tau} \operatorname{im}\tau \longrightarrow 0 \tag{2}$$

$$0 \longrightarrow \operatorname{im}\tau \xrightarrow{\imath} \ker\sigma \longrightarrow \ker\sigma/\operatorname{im}\tau \longrightarrow 0 \tag{3}$$

$$0 \longrightarrow \operatorname{im}\sigma \xrightarrow{\imath'} \ker\tau \longrightarrow \ker\tau/\operatorname{im}\sigma \longrightarrow 0 \tag{4}$$

where the unnamed arrows are the canonical projections. It turns out that $\operatorname{im}\sigma = \ker\tau$ and thus the sequence (4) reduces to

$$0 \longrightarrow \operatorname{im}\sigma \xrightarrow{\imath'} \ker\tau \longrightarrow 0$$

where $\imath'$ is the identity.

Since K is assumed to be a regular G-complex we can form the simplicial orbit space and canonical simplicial projection $\pi\colon K \to K/G$ induces a surjective chain map $\pi\colon C(K) \to C(K/G)$. It follows that the chain map σ factors through π, that is, there exists a unique chain map $\kappa\colon C(K/G) \to \operatorname{im}\sigma$ such that the diagram

$$\begin{array}{ccc} C(K) & & \\ \downarrow{\scriptstyle\pi} & \searrow^{\sigma} & \\ C(K/G) & \xrightarrow{\kappa} & \operatorname{im}\sigma \end{array}$$

commutes. It is evident that κ is surjective.

We have not only an identity involving $\operatorname{im}\sigma$, but also an identity involving $\operatorname{im}\tau$, namely we can conclude that $\operatorname{im}\tau = \ker\pi$. Thus we get the following two exact sequences

$$0 \longrightarrow \operatorname{im}\tau \xrightarrow{\omega\imath} C(K) \xrightarrow{\pi} C(K/G) \longrightarrow 0 \tag{5}$$

$$0 \longrightarrow \operatorname{im}\tau \xrightarrow{\imath} \ker\sigma \xrightarrow{\pi} D(K) \longrightarrow 0 \tag{6}$$

where $D(K)$ denotes the image of $\ker\sigma$ in $C(K/G)$ under the projection π, that is $D(K) := \pi(\ker\sigma)$. Note that the sequence (6) is essentially (3). Further it follows that the sequence

$$0 \longrightarrow D(K) \xrightarrow{\theta} C(K/G) \xrightarrow{\kappa} \operatorname{im}\sigma \longrightarrow 0 \tag{7}$$

is exact where θ denotes the inclusion $\theta\colon D(K) \hookrightarrow C(K/G)$.

Definition (Special Homology Groups)**.** The *special homology groups of the simplicial G-complex K* are the following four homology groups:[4]

$$H^\sigma(K) := H(\ker\sigma) \qquad \overline{H}^\sigma(K) := H(\operatorname{im}\sigma)$$

$$H^\tau(K) := H(\ker\tau) \qquad \overline{H}^\tau(K) := H(\operatorname{im}\tau)$$

Moreover we shall denote by $I(K)$ the homology groups of the chain complex $D(K)$.

Note that apparently the homology groups $\overline{H}^\sigma(K)$ and $H^\tau(K)$ are identical. That is we are essentially dealing only with three distinct special homology groups.

[4]Note that the definition of $\overline{H}^\sigma_k(K)$ and $\overline{H}^\tau_k(K)$ in this text differs from the way they are defined in [Yan60], where Yang defines $\overline{H}^\sigma_k(K) := H_k(\operatorname{im}\tau)$ and $\overline{H}^\tau_k(K) := H_k(\operatorname{im}\sigma)$! I have chosen this notation because I believe that this notation is more intuitive than the original one.

Passing to homology we get from the short exact sequences (1), (2), (5), (6) and (7) in turn the long exact homology sequences

$$\ldots \longrightarrow H_k^\sigma(K) \xrightarrow{\omega_*} H_k(K) \xrightarrow{\sigma_*} \overline{H}_k^\sigma(K) \longrightarrow H_{k-1}^\sigma(K) \xrightarrow{\omega_*} \ldots$$

$$\ldots \longrightarrow H_k^\tau(K) \xrightarrow{\omega'_*} H_k(K) \xrightarrow{\tau_*} \overline{H}_k^\tau(K) \longrightarrow H_{k-1}^\tau(K) \xrightarrow{\omega'_*} \ldots$$

$$\ldots \longrightarrow \overline{H}_k^\tau(K) \xrightarrow{\omega_* \imath_*} H_k(K) \xrightarrow{\pi_*} H_k(K/G) \longrightarrow \overline{H}_{k-1}^\tau(K) \xrightarrow{\omega_* \imath_*} \ldots$$

$$\ldots \longrightarrow \overline{H}_k^\tau(K) \xrightarrow{\imath_*} H_k^\sigma(K) \xrightarrow{\pi_*} I_k(K) \longrightarrow \overline{H}_{k-1}^\tau(K) \xrightarrow{\imath_*} \ldots$$

and

$$\ldots \longrightarrow I_k(K) \xrightarrow{\theta_*} H_k(K/G) \xrightarrow{\kappa_*} \overline{H}_k^\sigma(K) \longrightarrow I_{k-1}(K) \xrightarrow{\theta_*} \ldots$$

where the non-labeled homomorphisms are the appropriate connecting homomorphisms.

Finally we give an explicit description of the chain complex $D(K)$. Therefore denote for $0 \leq i \leq r$ by H_i the subgroups of G generated by $g_0^{[i]}$. Then those groups form a descending series

$$G = H_0 \supset H_1 \supset \ldots \supset H_{r-1} \supset H_r = 0$$

of subgroups of G.[5] We denote by $L_i := K^{H_i}$ the fixed point sets of those subgroups, which form then in turn an ascending series

$$L_0 \subset L_1 \subset \ldots \subset L_{r-1} \subset L_r = K$$

of regular simplicial G-complexes. Thus the orbit spaces L_i/G themself form simplicial complexes and we obtain an ascending series

$$L_0/G \subset L_1/G \subset \ldots \subset L_{r-1}/G \subset L_r/G = K/G$$

of simplicial subcomplexes of K/G.

For the chain complex one has now the explicit expression

$$D(K) = \sum_{k=0}^{r-1} C(L_i/G; Z_{[r-i]})$$

where Z_m denotes the subgroup of $\mathcal{R}$ with precisely m elements. Note that this sum on the right hand side is not a direct sum!

Note that since the coefficient group $\mathcal{R}$ is compact and since K is a finite simplicial complex all the homology groups introduced in this section are compact groups and the induced homomorphisms are continuous.

5. Special Coverings

The link between the algebraic aspect of the Čech homology groups and topological properties of spaces are inverse systems of simplicial spaces having the spaces as a limit. There are different ways in constructing such simplicial complexes and in our case we will use nerves of open coverings. Yang showed the existence of certain special open coverings which have nerves possessing just the right properties needed to apply the above constructed modified version of Smith-Theory.

In the following we assume that X is a compact G-space where G is a cyclic p-group as discussed before. We shall denote by F_i the fixed point set X^{H_i} of the subgroup H_i. We obtain an ascending sequence

$$X^G = F_0 \subset F_1 \subset \ldots \subset F_{r-1} \subset F_r = X$$

[5]Note that this describes the subgroup structure of G exhaustively

of closed subsets of X. Yang introduces in his article the following definition for a special covering.

Definition (Special Covering)**.** Let λ be a (self-indexed) covering of X. Then λ is said to be a *special covering* if it satisfies the following four conditions:

(1) The covering λ is finite and G-invariant.
(2) For every $U \in \lambda$ and $g \in G$ follows from $gU \neq U$ that $\overline{gU} \cap \overline{U} = \emptyset$.
(3) Let $U \in \lambda$ and $g \in G$. If there exist a $V \in \lambda$ which is not g-invariant, then from $U \cap V \neq \emptyset$ and $U \cap gV \neq \emptyset$ it follows that necessarily $gU = U$.
(4) For any H_i-invariant members $U_0, \ldots, U_k$ of λ with $U_0 \cap \ldots \cap U_k \neq \emptyset$, the intersection $U_0 \cap \ldots \cap U_k \cap F_i \neq \emptyset$, too.

Note that in the case of $G = C_p$ this definition is equivalent with the definition by Smith given in [Smi38].

Now the next two results show that special coverings are the proper tool to extend the modified Smith Theory to the Čech homology theory.

Proposition. *Let λ be a special covering of A in X. Then the action of G on X defines a regular action of G on the nerve K_λ.*

In other words this proposition means that we can apply the results from the previous section the the nerve K_λ of a special covering λ.

Proposition. *Let A be a closed G-invariant subset of X and α a covering of X. Then there exists an open special covering λ of X refining α.*

In other words the set of all open special coverings of X is cofinal in the set of all open coverings of X.[6] The Čech homology theory is defined with the help of inverse limits of homology groups directed by the sets of open coverings. Since passing to a cofinal subset of a directed set does not change the limit of an inverse system the above result makes it possible to reduce our attention in the following to special coverings if needed.

6. Extension to Čech Homology Theory

Still one of the best introduction to the Čech homology theory can be found in the classical book "Foundation of Algebraic Topology" by Eilenberg and Steenrod. [ES52]

Throughout this section X will be a compact G-space where G is a cyclic p-group. If λ is a special open covering of X, then action of G on X makes the nerve K_λ into a regular G-complex. Since λ is a finite cover it follows that K_λ is a finite complex and we can apply the results from Section 4 to the chain complex of ordered simplices of K_λ and its homology groups. The chain complexes and chain maps obtained this way depend of course on the choice of the covering λ. We shall indicate this dependency by using lower indices, that is we will denote those chain maps by σ_λ, τ_λ, ω_λ, $\imath_\lambda$, etc.

Let μ be another special covering of X and assume that $p_{\mu\lambda}\colon \mu \to \lambda$ is a refinement projection. It follows that we can assume without any loss of generality that p is G-equivariant. Then p induces chain maps $p_{\mu\lambda}\colon K_\mu \to K_\lambda$ and $p_{\mu\lambda}\colon K_\mu/G \to K_\lambda/G$ which are unique up to contiguity.[7]

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[6] Actually if one is precise, then the set of all open coverings is not a set but proper class. Nevertheless it is not a big problem to see over this technicality as there exists a technical trick to ship around this problem. [ES52]

[7] Recall that contiguous chain maps induce the same homomorphisms when passing to homology.

One verifies the following observations. First of all any G-equivariant refinement projection $p_{\lambda\mu}\colon \mu \to \lambda$ induces the chain maps

$$\begin{aligned} &p_{\lambda\mu}\colon \operatorname{im}\sigma_\mu \to \operatorname{im}\sigma_\lambda && p_{\lambda\mu}\colon \ker\sigma_\mu \to \ker\sigma_\lambda \\ &p_{\lambda\mu}\colon \operatorname{im}\tau_\mu \to \operatorname{im}\tau_\lambda && p_{\lambda\mu}\colon \ker\tau_\mu \to \ker\tau_\lambda \\ &p_{\lambda\mu}\colon C(K_\mu/G) \to C(K_\lambda/G) && p_{\lambda\mu}\colon D(K_\mu) \to D(K_\lambda) \end{aligned}$$

and furthermore the chain maps induced by the refinement projections commute with the chain maps introduced in Section 4, that is we have the equalities $p_{\lambda\mu}\sigma_\mu = \sigma_\lambda p_{\lambda\mu}$, $p_{\lambda\mu}\tau_\mu = \tau_\lambda p_{\lambda_\mu}$, $p_{\lambda\mu}\omega_\mu = \omega_\lambda p_{\lambda\mu}$, $p_{\lambda\mu}\imath_\mu = \imath_\lambda p_{\lambda\mu}$, $p_{\lambda\mu}\theta_\mu = \theta_\lambda p_{\lambda\mu}$, etc.

Since the above chain maps unique upto contiguity they induce unique homomorphisms when passing to homology. It follows that we obtain inverse systems of graded groups

$$\{H(K_\lambda), p_{\lambda\mu *}\}, \qquad \{H(K_\lambda/G), p_{\lambda\mu *}\}, \qquad \{H^\sigma(K_\lambda), p_{\lambda\mu *}\},$$
$$\{\overline{H}^\sigma(K_\lambda), p_{\lambda\mu *}\}, \qquad \text{etc.}$$

directed by the set of all special open covering of X. Important for the extension of the modified Smith Theory to Čech homology theory is the following result.

Proposition. *We have the two isomorphisms of graded groups*

$$\check{H}(X) \cong \varprojlim H(K_\lambda) \qquad \textit{and} \qquad \check{H}(X/G) \cong \varprojlim H(K_\lambda/G)$$

where the limits are taken over the inverse systems as described above.

Here the first isomorphism is a direct consequence of the fact that the special open coverings of X are cofinal in the set of all open coverings of X. The latter isomorphism needs more work until it is verified.

It turns out that the isomorphism $\check{H}(X/G) \cong \varprojlim H(K_\lambda/G)$ can be choosen such that the homomorphism $\pi_*\colon \check{H}(X) \to \check{H}(X/G)$ which is induced by the canonical projection onto the orbit space $\pi\colon X \to X/G$ agrees with the homomorphism which can be obtained from the chain maps $\pi_\lambda\colon C(K_\lambda) \to C(K_\lambda/G)$ as defined in Section 4.

Definition (Special Homology Groups)**.** The *special homology groups* of the compact G-space X are the four inverse limits

$$\begin{aligned} H^\sigma(X) &:= \varprojlim H^\sigma(K_\lambda) & \overline{H}^\sigma(X) &:= \varprojlim \overline{H}^\sigma(K_\lambda) \\ H^\tau(X) &:= \varprojlim H^\tau(K_\lambda) & \overline{H}^\tau(X) &:= \varprojlim \overline{H}^\tau(K_\lambda) \end{aligned}$$

where the limit is taken over the limiting systems as described before. Moreover we define $I(X)$ to be the limit of the inverse system $\{I(K_\lambda), p_*\}$.

Note that homology groups introduced in this section are again all compact and apparently the homology groups $\overline{H}^\sigma(X)$ and $H^\tau(X)$ are identical. Furthermore the homomorphisms introduced in Section 4 define in a natural way maps of inverse systems and their limits define continuous homomorphisms between the above homology groups. It follows from the long exact homology sequences of Section 4 and from the the naturality of the connecting homomorphism that the following long homology sequences are exact:

$$\ldots \longrightarrow H_k^\sigma(X) \xrightarrow{\omega_*} \check{H}_k(X) \xrightarrow{\sigma_*} \overline{H}_k^\sigma(X) \longrightarrow H_{k-1}^\sigma(X) \xrightarrow{\omega_*} \ldots$$

$$\begin{array}{llllllllll}
\dots \longrightarrow H^\tau_k(X) & \xrightarrow{\omega'_*} & \check H_k(X) & \xrightarrow{\tau_*} & \overline{H}^\tau_k(X) & \longrightarrow H^\tau_{k-1}(X) & \xrightarrow{\omega'_*} \dots \\
\dots \longrightarrow \overline{H}^\tau_k(X) & \xrightarrow{\omega_* \imath_*} & \check H_k(X) & \xrightarrow{\pi_*} & \check H_k(X/G) & \longrightarrow \overline{H}^\tau_{k-1}(X) & \xrightarrow{\omega_* \imath_*} \dots \\
\dots \longrightarrow \overline{H}^\tau_k(X) & \xrightarrow{\imath_*} & H^\sigma_k(X) & \xrightarrow{\pi_*} & I_k(X) & \longrightarrow \overline{H}^\tau_{k-1}(X) & \xrightarrow{\imath_*} \dots \\
\dots \longrightarrow I_k(X) & \xrightarrow{\theta_*} & \check H_k(X/G) & \xrightarrow{\kappa_*} & \overline{H}^\sigma_k(X) & \longrightarrow I_{k-1}(X) & \xrightarrow{\theta_*} \dots
\end{array}$$

Here the non-labeled homomorphisms are again the appropriate connecting homomorphisms.

7. Extension to p-Adic Transformation Groups

Recall the notation and resluts of Section 3 about p-adic integers. The key to the extension of the so far developed Smith Theory to the action of the group of the p-adic integers on a compact space X is the following observation:

Recall that the subgroups $\mathbf{Z}_i$ form an open neighboorhod system of the identity element of $\mathcal{Z}_p$. It follows that the limit of the inverse system $\{X/\mathbf{Z}_i, \pi_i^j\}$ (where $\pi_i^j\colon X/\mathbf{Z}_j \to X/\mathbf{Z}_i$ denote the canonical projections) is homeomorphic to X and that the inverse limit of the canonical projections $\pi_i\colon X \to X/\mathbf{Z}_i$ defines a homeomorphism

$$X \cong \varprojlim X/\mathbf{Z}_i.$$

The action of $\mathcal{Z}_p$ induces in a natural way an action of $\mathbb{Z}_{[i]}$ on $X/\mathbf{Z}_i$. Since $\mathbb{Z}_{[i]}$ is a cyclic p-group and since $X/\mathbf{Z}_i$ is a compact space we can apply the results of the previous section in this case. From the continuity property of the Čech homology theory follows that the projections $\pi_i\colon X \to X/\mathbf{Z}_i$ define an isomorphism

$$\check H(X) \cong \varprojlim \check H(X/\mathbf{Z}_i)$$

We use this fact to extend the so far developed Smith Theory to p-adic actions in a similar fashion as we have extendet it to the Čech homology theory in the previous section.

In order to apply the results of the previous section we fix an element $\hat g \in \mathcal{Z}_p \setminus \mathbf{Z}_1$. We set $g_i := h_i(\hat g)$ for each $i \in \mathbb{N}$. Then g_i is a generator of the cyclic p-group $\mathbb{Z}_{[i]}$ which is acting on $X/\mathbf{Z}_i$. Now the results of the previous section apply. We shall emphasis the dependecy of the so obtained homomorphism on $\mathbb{Z}_{[i]}$ by adding "i" as an index to those maps. That is we have

$$\sigma_{i*} = \operatorname{id} + g_{i*} + g_{i*}^2 + \dots + g_{i*}^{[i]-1} \qquad \text{and} \qquad \tau_{i*} = \operatorname{id} - g_{i*}$$

and further ω_{i*}, ω'_{i*}, $\imath_{i*}$, $\imath'_{i*}$, π_{i*}, θ_{i*}, etc.

Note that the algebraic structure of the homomorphism σ_{i*} is in an essential way dependent on i, whereas the algebraic structure of τ_{i*} is independent of i. This algebraic difference will foil our attempt to extend the theory developed so far in its full extend to the new environment.

We shall use the following notation: if λ_i is a special covering of $X/\mathbf{Z}_i$, then we denote for $j \geq i$ by λ_j the covering $\lambda_j := (\pi_i^j)^{-1}(\lambda_i)$ indexed by λ_i. It follows that the projection $\pi_i^j\colon X/\mathbf{Z}_j \to X/\mathbf{Z}_i$ dose induce the homomorphisms

$$\pi^j_{i*}\colon H^\tau(K_{\lambda_j}) \to H^\tau(K_{\lambda_i}), \qquad \pi^j_{i*}\colon \overline{H}^\tau(K_{\lambda_j}) \to \overline{H}^\tau(K_{\lambda_i}),$$
$$\pi^j_{i*}\colon \overline{H}^\sigma(K_{\lambda_j}) \to \overline{H}^\sigma(K_{\lambda_i}),$$

but π_i^j does *not* induce a homomorphism

$$\pi^j_{i*}\colon H^\sigma(K_{\lambda_j}) \to H^\sigma(K_{\lambda_i}).$$

Now we can form the inverse systems

$$\{\overline{H}^{\sigma}(X/\mathbf{Z}_i), \pi_*\}, \quad \{\overline{H}^{\tau}(X/\mathbf{Z}_i), \pi_*\} \quad \text{and} \quad \{H^{\tau}(X/\mathbf{Z}_i), \pi_*\}$$

directed by the natural numbers $\mathbb{N}$. But since π_i^j does not induce a homomorphism $H^{\sigma}(K_{\lambda_j}) \to H^{\sigma}(K_{\lambda_i})$ the collection $\{H^{\sigma}(X/\mathbf{Z}_i), \pi_*\}$ does not form a inverse system.

Therefore we can only define three (and not four) special homology groups:

Definition (Special Homology Groups for the p-Adic Integers)**.** Let X be a compact $\mathcal{Z}_p$-space. Then the *special homology groups* of X are defined to be the following inverse limits

$$\overline{H}^{\sigma}(X) := \varprojlim \overline{H}^{\sigma}(X/\mathbf{Z}_i)$$
$$H^{\tau}(X) := \varprojlim H^{\tau}(X/\mathbf{Z}_i)$$
$$\overline{H}^{\tau}(X) := \varprojlim \overline{H}^{\tau}(X/\mathbf{Z}_i)$$

where the limits are taken over the limiting systems as described before.

Note that we can not extend the definition of the group I_k since it can be shown that the projection π_i^j does *not* induce a homomorphism $D_k(K_{\lambda_j}) \to D_k(K_{\lambda_i})$.

Apparently $\overline{H}^{\sigma}(X) = H^{\tau}(X)$ and therefore we deal essentially with only two distinct special homology groups in the p-adic case, namely $H^{\tau}(X)$ and $\overline{H}^{\tau}(X)$.

When passing to the limit then the collection of homomorphisms $\{\tau_{i*}\}$, $\{\omega'_{i*}\}$, $\{(\omega_i\imath_i)_*\}$ and $\{\pi_{i*}\}$[8] define the homomorphism

$$\tau_*\colon \check{H}(X) \to \overline{H}^{\tau}(X), \qquad \omega'_*\colon H^{\tau}(X) \to \check{H}(X),$$
$$(\omega\imath)_*\colon \overline{H}^{\tau}(X) \to \check{H}(X), \qquad \pi_*\colon \check{H}(X) \to \check{H}(X/\mathcal{Z}_p).$$

Note that for the remaining homomorphisms the above extension to the p-adic case cannot be made. Since π_{i*}^j and σ_{j*} do not commute we cannot define the homomorphism σ_* and since for a similar reason π_{i*}^j does not commute with κ_{j*} we we *cannot* define the homomorphism κ_* for the p-adic case. Moreover we have already noted before that the groups $H^{\sigma}(X)$ and $I(X)$ cannot be defined in the p-adic case and therefore we *cannot* define any of the homomorphisms $\omega_*\colon H^{\sigma}(X) \to \check{H}(X)$, $\imath_*\colon \overline{H}^{\tau}(X) \to H^{\sigma}(X)$ and $\theta_*\colon I(X) \to \check{H}(X)$ in the p-adic case.

Thus only two of the five long exact homology sequences from the previous section can be carried over to the p-adic case. We get the following result.

Proposition. *The two long homology sequences*

$$\ldots \longrightarrow H_k^{\tau}(X) \xrightarrow{\omega'_*} \check{H}_k(X) \xrightarrow{\tau_*} \overline{H}_k^{\tau}(X) \longrightarrow H_{k-1}^{\tau}(X) \xrightarrow{\omega'_*} \ldots$$
$$\ldots \longrightarrow \overline{H}_k^{\tau}(X) \xrightarrow{(\omega\imath)_*} \check{H}_k(X) \xrightarrow{\pi_*} \check{H}_k(X/\mathcal{Z}_p) \longrightarrow \overline{H}_{k-1}^{\tau}(X) \xrightarrow{(\omega\imath)_*} \ldots$$

are exact where the unnamed arrows are the apparent connecting homomorphisms.[9]

[8] Note that Yang claimed in mistake in his article that $\pi_i\pi_i^j \neq \pi_i^j\pi_j$ and thus he could not extend the definition of π_* to a homomorphism $\pi_*\colon \check{H}(X) \to \check{H}(X/\mathcal{Z}_p)$.

[9] This is Lemma 5.2 and Corollary 5.4 in Yang's paper, though Yang has a slightly weaker result for the second sequence.

8. Applying Homology Dimension

Definition (Homology Dimension for Compact Spaces). Let X be a compact space and let $\mathcal{G}$ be a compact abelian group. Let $n \geq -1$ be an integer. Then the space X is said to have *homology dimension* $\leq n$ (with respect to the coefficient group $\mathcal{G}$), if for every compact pair (M, N) with $M \subset X$ and $k > n$ holds $\check{H}_k(M, N; \mathcal{G}) = 0$. If X is of homology dimension $\leq n$ but not of homology dimension $\leq n-1$, then X is said to have *homology dimension* n. If X is not of homology dimension $\leq n$ for any integer, then X is said to have *infinite homology dimension*. In symbols we denote the fact that X has the homology dimension n or infinite (with respect to $\mathcal{G}$) by $\operatorname{hd}_{\mathcal{G}} X := n$ or $\operatorname{hd}_{\mathcal{G}} X := \infty$ respectively.

It follows that $\operatorname{hd}_{\mathcal{G}} X \leq \operatorname{hd}_{\mathcal{R}} X$ for any compact coefficient group $\mathcal{G}$. [Ale47] Therefore the homology dimension works nicely together with Yang's choice of $\mathcal{R}$ as the coefficient groups in his modified version of Smith Theory. We therefore omit $\mathcal{R}$ in the notation of homology dimension, that is, we will write $\operatorname{hd} X$ instead of $\operatorname{hd}_{\mathcal{R}} X$.

If $\dim X$ denotes the Lebesque covering dimension of X, then it follows that $\operatorname{hd} X \leq \dim X$ and equality holds if $\dim X$ is finite. [Ale47] If A is a closed subset of X, then $\operatorname{hd}_{\mathcal{G}} A \leq \operatorname{hd}_{\mathcal{G}} X$ for any compact coefficient group $\mathcal{G}$. In particular $\operatorname{hd} A \leq \operatorname{hd} X$.

We can use the homology dimension of X in order to improve the results of the previous two sections.

Assume first that X is a compact G space where G is a cyclic p-group. Assume further that $\operatorname{hd} X \leq n$. Then by definition $\check{H}_k(X) = 0$ for $k > n$ and it follows that the special homology groups $H_k^\sigma(X)$, $\overline{H}_k^\sigma(X)$, $H_k^\tau(X)$ and $\overline{H}_k^\tau(X)$ are all trivial for $k > n$, too. In particular the homomorphisms

$$\omega_*\colon H_k^\sigma(X) \to \check{H}_k(X) \qquad \text{and} \qquad \omega'_*\colon H_k^\tau(X) \to \check{H}_k(X)$$

are monomorphisms for $k \geq n$.

In the case that X is a compact $\mathcal{Z}_p$ space Yang shows the following: assume that the homology dimension of the fixed point sets $X^{\mathbf{Z}_i}$, $i \in \mathbb{N}$ are bounded by some integer n, then the groups $I_k(X/\mathbf{Z}_i)$, $i \in \mathbb{N}$ are trivial for $k > n$ and as a consequence the homomorphisms

$$\imath_{i*}\colon \overline{H}_k^\tau(X/\mathbf{Z}_i) \to H_k^\sigma(X/\mathbf{Z}_i) \qquad \text{and} \qquad \kappa_{i*}\colon \check{H}_{k+1}(X/\mathcal{Z}_p) \to \overline{H}_{k+1}^\sigma(X/\mathbf{Z}_i)$$

are monomorphisms for $k \geq n$ and even isomorphisms for $k > n$.

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