SHORT COURSE ON ALGEBRAIC TOPOLOGY

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Abstract. This short course on Algebraic Topology shall give an introduction to some essential concepts of this field of mathematics: the fundamental group and homology theories. As an example we will show how those results are used to prove that the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$, which is not possible to prove in a purely topological way.

1. Introduction

In short Algebraic Topology is the study of algebraic invariants attached to topological spaces. As an example consider the following theorem:

Theorem 1. *Let $n \neq m$ be two non-negative integers. Then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.*

At first this result seems to be intuitive: an n-dimensional space should not be homeomorphic to an m-dimensional space. But the dimension of topological spaces is not an invariant of continuous maps. It is very easy to construct examples of continuous maps which lower the dimension, like the projection

$$\pi\colon \mathbb{R}^n \to \mathbb{R}^m, (x_1, \dots, x_n) \mapsto (x_1, \dots, x_m), \qquad m < n.$$

There are even continuous maps which rise the dimension in a way which is very counterintuitive. For example consider the closed unit interval $I = [0, 1]$. Then there exist continuous maps

$$\varphi\colon I \to I^2.$$

which are surjective. The first construction of such a map was was done by Peano in 1890, see for example [Sag94].

So why should it be intuitive after all that there exists no homeomorphism $\mathbb{R}^n \to \mathbb{R}^m$ in the case $n \neq m$? And infact there is no proof known for the above theorem which is of purely topological nature. The necessary tools to tackle the proof are provided by algebraic topology:

For $q \in \mathbb{Z}$ we denote by $\tilde{H}_q\colon \mathbf{Top} \to \mathbf{Ab}$ the *q-th reduced singular homology functor* from the category **Top** of topological spaces and continuous maps to the category **Ab** of abelian groups and group homomorphisms. If one calculates the values of this functor for the n-sphere S^n one obtains

$$\tilde{H}_q(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Using this information we can now prove the above theorem.

Proof of Theorem 1. Assume towards a contradiction that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ for $n \neq m$. Then also their one point compactifications S^n and S^m would be homeomorphic and this would imply that $\tilde{H}_n(S^n) \cong \mathbb{Z}$ is isomorphic to $\tilde{H}_n(S^m) = 0$. But this silly and yields therefore a contradiction. □

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Observe that the main work in proving Theorem 1 lies in the result (1). We shall in the following introduce two important examples of working tools used by algebraic topologists.

2. The Fundamental Group

A *topological space with basepoint* is a pair $X = (X, x_0)$ consisting of a topological space X and a fixed element $x_0 \in X$, which is called the *basepoint* of X. A *map* $f\colon X \to Y$ of topological spaces with basepoints is a continuous map such that $f(x_0) = y_0$.

We want to construct a functor $\pi_1\colon \mathbf{Top}_0 \to \mathbf{Grp}$ from the category $\mathbf{Top}_0$ of topological spaces with basepoint and maps between them to the category $\mathbf{Grp}$ of groups and homomorphisms. The construction is done as follows.

Let X be a topological space. A *path* in X is a continuous map $\alpha\colon I \to X$ from the unit interval I to the space X. The points $\alpha(0)$ and $\alpha(1)$ are called the *endpoints* of α. The path α is *closed* if $\alpha(0) = \alpha(1)$. Let $x_0 \in X$. A *closed path at* x_0 is a closed path α in X with endpoints x_0.

If α and β are two paths in X with the same endpoints (that is $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$), then we say that α is *homotopic* to β if there exists a continuous map $F\colon I \times I \to X$ – called *homotopy* – such that

1. $F(0, t) = \alpha(0)$ and $F(1, t) = \alpha(1)$ for every $t \in I$, and
2. $F(x, 0) = \alpha(x)$ and $F(x, 1) = \beta(x)$ for every $x \in I$.

If α and β are homotopic paths then we denote this fact by $\alpha \simeq \beta$. It follows that "$\simeq$" is an equivalence relation. If α is a path in X, then we denote by $[\alpha]$ the equivalence class of all paths in X homotopic to α.

Now let X be a topological space with basepoint x_0. If α and β are two closed paths at x_0 then we can define the *product* $\alpha\beta$ to be the map $\alpha\beta\colon I \to X$ given by

$$(\alpha\beta)(t) := \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq 1/2, \\ \beta(2t-1) & \text{if } 1/2 < t \leq 1. \end{cases}$$

It follows that the product $\alpha\beta$ is again a closed path at x_0. Furthermore one can show that if α' and β' are closed paths in X at x_0 such that $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$, then also $\alpha\beta \simeq \alpha'\beta'$. Thus

$$[\alpha][\beta] := [\alpha\beta]$$

gives a well defined product of equivalence classes of closed paths at x_0. It follows that the set of all equivalence classes of closed paths at x_0 has the structure of a group.[1]

Definition 2 (Fundamental Group)**.** Let X be a topological space with basepoint x_0. Then the *fundamental group X with basepoint x_0* is the group

$$\pi_1(X, x_0) := \{[\alpha] : \alpha \text{ is a closed path at } x_0\}.$$

Note that the fundamental group of a space depends very much on the choice of the base point! Note further that in general the group $\pi_1(X, x_0)$ is not abelian.

If $f\colon X \to Y$ is a map of topological spaces with basepoints x_0 and y_0 then for any closed path α at x_0 the composite $f \circ \alpha$ is a closed path at y_0. It follows that if α' is another closed path at x_0 homotopic to α, then $f \circ \alpha'$ is homotopic to $f \circ \alpha$. Thus

$$f_*([\alpha]) := [f \circ \alpha]$$

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[1] The equivalence class $[\imath]$ of the constant path $\imath\colon I \to X, t \mapsto x_0$ is the identity element. If α is a closed path at x_0 then so is $\alpha'\colon I \to X, t \mapsto \alpha(1-t)$ and the equivalence class $[\alpha']$ is the inverse element of $[\alpha]$.

gives a well defined map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. It is not difficult to verify that f_* is a group homomorphism.

Definition 3. The homomorphism f_* is called the homomorphism *induced* by f.

The following observations can be made. If $f\colon X \to Y$ and $g\colon Y \to Z$ are maps of topological spaces with basepoint then

$$g_* \circ f_* = (g \circ f)_*$$

and the identity map $\mathrm{id}\colon X \to X$ induces the indentity map $\mathrm{id}\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$. Collecting these results we obtain the following proposition.

Proposition 4. *There exists a covariant functor $\pi_1\colon \mathbf{Top}_0 \to \mathbf{Grp}$ from the category $\mathbf{Top}_0$ of topological spaces with basepoint to the catergory $\mathbf{Grp}$ of groups which assigns every topological space X with basepoint x_0 the fundamental group $\pi_1(X, x_0)$ and which assigns every map $f\colon X \to Y$ of topological spaces with basepoint the induced homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.*

In the special case that X is a path connected space it follows that the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_0')$ are isomorphic for any basepoints x_0 and x_0'. In particular it follows that if X and Y are path connected spaces which are homeomorphic, then $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are isomorphic groups. As an application we can prove the following special case of Theorem 1.

Proposition 5. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.*

Proof. Assume towards a contradiction that there exists a homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^n$. Then the restriction of f to $\mathbb{R}^2 \setminus \{0\}$ yields a homeomorphism from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{R}^n \setminus \{f(0)\}$.

In the case that $n = 1$ this yields a contradiction since $\mathbb{R}^2 \setminus \{0\}$ is connected but $\mathbb{R}^1 \setminus \{0\}$ is not connected.

For $n > 2$ use the following reasoning: let x_0 be a basepoint of $\mathbb{R}^2 \setminus \{0\}$. Then there exist closed paths at x_0 which are not homotopic to the constant x_0 path. Thus $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$ must be a non-trivial group.[2] On the other hand any closed path in $\mathbb{R}^n \setminus \{f(0)\}$ at y_0 is homotopic to the constant y_0 path and therefore $\pi_1(\mathbb{R}^n \setminus \{f(0)\}, y_0)$ is the trivial group. Therefore $\mathbb{R}^2 \setminus \{0\}$ and $\mathbb{R}^n \setminus \{f(0)\}$ cannot be homeomorphic in this remaining case. ☐

3. Homology Theories

A *(topological) pair* (X, A) consists of a topological space X and a subspace $A \subset X$. It is convenient to abbreviate the pair $(X, \emptyset)$ by X. We may add topological properties to a pair (X, A) which then implies that both X and A satisfis this property. For example a closed pair (X, A) is a pair where A is a closed subset of X. Likewise a compact[3] pair (X, A) consists of a compact space X and a compact subspace A.

A map $f\colon (X, A) \to (Y, B)$ of pairs is a continuous map $f\colon X \to Y$ such that $f(A) \subset B$.

In the following we will describe another tool of the algebraic topologist: homology theories. Those theories can be defined for categories of pairs and maps between them which are *admissible categories for homology theory* in the sense of Eilenberg and Steenrod [ES52]. The pairs and maps in such a category are called *admissible.*

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[2] The precise result is that $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0) \cong \mathbb{Z}$.

[3] In this text compactness – and therefore also localy compactness – shall always imply the Hausdorff condition.

Examples of such categories are the following:

1. The category $\mathcal{A}_1$ of all pairs (X, A) and all maps of such pairs. This is the largest admissible category.
2. The category $\mathcal{A}_C$ of all compact pairs (X, A) and all maps of such pairs.
3. The category $\mathcal{A}_{LC}$ of all pairs (X, A) such that X is a locally compact space, A a closed subset of X and all proper[4] maps between them.

We make the following conceptual definition: A homology theory $H = (H, f_*, \partial)$ on an admissible category $\mathcal{A}$ is a triple consisting of three functions:

1. The first function $H_q(X, A)$ is defined for every admissible pair (X, A) and $q \in \mathbb{Z}$. Its value is an abelian group and it is called the *q-dimensional relative homology group of X modulo A.*
2. The second function is defined for each admissible map $f\colon (X, A) \to (Y, B)$ and each $q \in Z$ and assigns each such pair a homomorphism
$$f_*\colon H_q(X, A) \to H_q(Y, B).$$
It is called the homomorphism *induced* by f.
3. The third function $\partial(q, X, A)$ is defined for every admissible pair (X, A) and $q \in \mathbb{Z}$. Its value is a homomorphism
$$\partial\colon H_q(X, A) \to H_{q-1}(A)$$
and is called the *boundary operator.*[5]

These three functions are required to staisfy the following seven *Eilenberg–Steenrod* axioms for homology:

Axiom 1. *If f is the identity then f_* is the identity, too.*

Axiom 2. $(g \circ f)_* = g_* \circ f_*$.

Axiom 3. $\partial \circ f_* = (f|A)_* \circ \partial$, *where $f|A$ denotes the restriction of f to A.*

Axiom 4 (Exactness Axiom)**.** *If (X, A) is admissible and $i\colon A \to X$ and $j\colon X \to (X, A)$ are the inclusion maps, then the descending sequence of groups and homomorphisms*

$$\dots \xrightarrow{\partial} H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \xrightarrow{i_*} \dots \qquad (2)$$

is exact.[6]

Axiom 5 (Homotopy Axiom)**.** *If the admissible maps $f_0, f_1\colon (X, A) \to (Y, B)$ are homotopic in $\mathcal{A}$,[7] then the induced homomorphisms*

$$f_{0*}, f_{1*}\colon H_q(X, A) \to H_q(Y, B)$$

coincide for every $q \in \mathbb{Z}$.

Axiom 6 (Excision Axiom)**.** *If U is an open subset of X whose closure $\overline{U}$ is contained in the interior* Int A *of A and if the the inclusion map $i\colon (X \setminus U, A \setminus U) \to (X, A)$ is admissible, then*

$$i_*\colon H_q(X \setminus U, A \setminus U) \to H_q(X, A)$$

is an isomorphism for every $q \in \mathbb{Z}$.

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[4] A continuous map is called *proper* if every preimage of a compact set is compact.

[5] Note that $H_{q-1}(A)$ is an abreviation for $H_{q-1}(A, \emptyset)$.

[6] A sequence is called exact if the image of every homomorphism in the sequence is equal to the kernel of the homomorphism next in the sequence.

[7] f_0 and f_1 are *homotopic in $\mathcal{A}$* if there exists an admissible homotopy $F\colon (X, A) \times I \to (Y, B)$ from f_0 to f_1. Here $(X, A) \times I$ denotes the pair $(X \times I, A \times I)$.

Axiom 7 (Dimension Axiom)**.** *If P is an admissible space consisting of a single point, then $H_q(P) = 0$ for every non-zero $q \in \mathbb{Z}$.*

The first two axioms say that $H\colon \mathcal{A} \to \mathbf{Ab}$ is a covariant functor from the admissible category $\mathcal{A}$ to the category $\mathbf{Ab}$ of abelian groups and homomorphisms.

4. Singular Homology Theory

As a concrete example of a homology theory we shall outline the construction of the *singular homology theory.* It can be defined for the admissible category $\mathcal{A}_1$.[8] We do this by first constructing singular chain complexes and chain maps between them and then – as the homological algebraist says – passing to homology.

Let q be a non-negative integer. The *standard q-simplex* Δ^q is defined to be the subspace of $\mathbb{R}^{q+1}$ given by

$$\Delta^q := \{(t_0, \ldots, t_q) \in \mathbb{R}^{q+1} : t_0 + \ldots + t_q = 1 \text{ and } t_i \geq 0 \text{ for all } i\}.$$

Let X be an arbitrary topological space. A continuous map

$$\sigma\colon \Delta^q \to X.$$

is called a *singular q-simplex* in the space X.[9] The free group generated by all singular q-simplices in X is denoted by $S_q(X)$.[10]

For every integer $q > 0$ we define $q + 1$ face operators

$$d_i\colon \Delta^{q-1} \to \Delta^q, (t_0, \ldots, t_{q-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{q-1}), \qquad (0 \leq i \leq q).$$

Then for every $0 \leq i \leq q$ the map $\sigma \circ d_i\colon \Delta^{q-1} \to X$ is a singular $(q-1)$-simplex in X and called the *i-th face* of σ. With the help of the face operators we can define a homomorphism $\partial_q\colon S_q(X) \to S_{q-1}(X)$ by defining it on the generators of S_q: for every singular q-simplex σ we set

$$\partial(\sigma) := \sum_{i=0}^{q} (-1)^i (\sigma \circ d_i).$$

It is not difficult to verify that $\partial_q \circ \partial_{q+1} = 0$ is the trivial homomorphism. We call the collection

$$S(X) := \{S_q(X), \partial_q\}$$

the *singular chain complex*[11] *of X.*

Assume that $f\colon X \to Y$ is a continuous map. Then for every singular q-simplex σ in X we have that $f(\sigma) := f \circ \sigma$ is a singular q-simplex in Y. This way we obtain a collection $f = \{f_q\}$ of maps

$$f_q\colon S_q(X) \to S_q(Y).$$

It is not difficult to verify that $\partial_q \circ f_q = f_{q-1} \circ \partial_q$ for ever integer q. In the language of homological algebra this means that $f\colon S(X) \to S(Y)$ is a chain map (of degree 0).

If (X, A) is a pair, then the inclusion $i\colon A \to X$ defines for every integer q a monomorphism $i_q\colon S_q(A) \to S_q(X)$ and we get short exact sequences

$$0 \longrightarrow S_q(A) \xrightarrow{i_q} S_q(X) \xrightarrow{\pi_q} S_q(X, A) \longrightarrow 0 \tag{3}$$

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[8] Not every homology theory can be defined for this large category.

[9] The word "singular" shall express the idea that σ needs not to be in any way a nice embedding of Δ^q into X, it might have "singularities" where its image does not look like an simplex at all. The only requirement we have on σ is that it is a continuous map.

[10] Since there exists no singular q-simplices for $q < 0$ the groups $S_q(X)$ are trivial for $q < 0$.

[11] A *chain complex of groups* $C = (C_q, \partial_q)$ consists of a collection of groups C_q and homomorphism $\partial_q\colon C_q \to C_{q-1}$ indexed by integers such that $\operatorname{im} \partial_{q+1} \subset \ker \partial_q$ for every $q \in \mathbb{Z}$. A *chain map* $f\colon C \to C'$ (of degree 0) between two chain complexes consists of a collection of homomorphism $f_q\colon C_q \to C'_q$ such that $\partial'_q \circ f_q = f_{q-1} \circ \partial_q$.

where $S_q(X, A)$ denote the quotient group $S_q(X)/S_q(A)$ and where π_q denote the canonical projection. It follows that there exist unique maps $\partial_q\colon S_q(X, A) \to S_{q-1}(X, A)$ such that $\partial_q \circ \pi_q = \pi_{q-1} \circ \partial_q$. Therefore we obtain a chain complex

$$S(X, A) := \{S_q(X, A), \partial_q\}.^{12}$$

This chain complex is called the *relative singular chain complex of X modulo A*. Furthermore, it follows that if $f\colon (X, A) \to (Y, B)$ is a map of pairs then there exist unique homomorphisms

$$f_q\colon S_q(X, A) \to S_q(Y, B)$$

such that $\pi_q \circ f_q = f_q \circ \pi_q$ for every integer q. It follows that the collection $f = \{f_q\}$ defines a chain map $f\colon S(X, A) \to S(Y, B)$.

We note the following standard definition and results from homological algebra: Given a chain complex $C = (C_q, \partial_q)$ we denote by $Z_q(C) := \ker(\partial_q)$ the *group of q-cycles* and by $B_q(C) := \operatorname{im}(\partial_{q+1})$ the *group of q-boundaries*. The subgroup $B_q(C)$ is normal[13] in $Z_q(C)$ and we can define the *q-th homology group $H_q(C)$* of the chain complex C to be the quotient group

$$H_q(C) := Z_q(C)/B_q(C).$$

Furthermore a chain map $f\colon C \to C'$ gives always rise to *induced homomorphisms*

$$f_*\colon H_q(C) \to H_q(C') \tag{4}$$

when passing to homology.

Using this notation and results we can now define for any pair (X, A) the *relative singular homology groups of X modulo A* $H_q(X, A)$ to be the homology groups of the singular chain complex $S(X, A)$, that is

$$H_q(X, A) := H_q(S(X, A)).$$

If $f\colon (X, A) \to (Y, B)$ is a map of pairs, then the *homomorphism induced by f*, that is

$$f_*\colon H_q(X, A) \to H_q(Y, B)$$

is defined to be the homomorphism $f_*\colon H_q(S(X, A)) \to H_q(S(Y, B))$ as defined in (4). Finally the boundary homomorphism $\partial\colon H_q(X, A) \to H_{q-1}(A)$ is defined with the help of a standard result from homological algebra: when this result is applied to the short exact sequence of chain complexes

$$0 \longrightarrow S(A) \xrightarrow{i} S(X) \xrightarrow{j} S(X, A) \longrightarrow 0, \tag{5}$$

which is derived from (3),[14] then this result states that there exists a boundary homomorphism $\partial\colon H_q(X, A) \to H_{q-1}(A)$ such that the long homology sequence (2) is exact.

It can then be shown that the above functions indeed yield a homology theory $H = (H, f_*, \partial)$ on $\mathcal{A}_1$ which satisfies all Eilenber-Steenrod axioms for homology. This homology theory is called the *singular homology theory*.

It shall be noted that even though the groups $S_q(X, A)$ and with it the groups $B_q(S(X, A))$ and $Z_q(S(X, A))$ are usually extremely large it turns out that the

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[12] Observe that we can identify $S(X)$ and $S(X, \emptyset)$.

[13] We are dealing with abelian groups after all!

[14] Note that the homomorphism π_q is the same as the homomorphism j_q induced by the inclusion $X \to (X, A)$ of pairs.

homology groups $H_q(X, A)$ can be very small and simple in their structure. For example it is not difficult to compute for the n-sphere S^n, $n \geq 0$, that

$$H_q(S^n) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 0 \text{ and } q = 0, \\ \mathbb{Z} & \text{if } n > 0 \text{ and } q = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

From this result one can prove then Theorem 1 in the same way as we have done in the introduction.

5. Some Noteworthy Literature

One excllent introduction to Algebraic Topology can be found in the classical book "Foundations of Algebraic Topology" by Eilenberg and Steenrod [ES52].

Another very good book on this field of mathematics is Spanier's "Algebraic Topology" [Spa66].

The above mentioned books are excellent resources but not necessarily intended for a newcommer to this subject. This gap is filled by the book "Algebraic Topology" by Hatcher [Hat02]. It contains a lot of examples and many illustrations to give a better understanding for the theoretical concepts. The book is even freely available for download on the internet (`http://www.math.cornell.edu/~hatcher/`).

Peter May gives in "A Concise Course in Algebraic Topology" [May99] a very beautiful introduction to Algebraic Topology from a modern point of view. Also this book can be obtained freely from the authors homepage at `http://www.math.uchicago.edu/~may/`.

Another introduction to Algebraic Topology, this time with a strong flavoring in smooth manifolds, can be found in Glen Bredon's book "Topology and Geometry" [Bre93]. Furthermor William Massey's book "A Basic Course in Algebraic Topology" [Mas91] tries to emphaisis whenever possible the geometric motivation behind the various concepts introduced.

If one is interested in how Algebraic Topology did develop, then the book "Basic Concepts of Algebraic Concepts" by Fred Croom [Cro78] is noteworthy as it includes a lot of historical notes alongside with the developement of the theory. Finally Jean Dieudonné has written a book which purely focuses on the historical development of Algebraic Topology during the period 1900 to 1960 [Die98].

References

[Bre93] Glen E. Bredon. *Topology and Geometry*. Graduate texts in mathematics. Springer–Verlag, 1993.
[Cro78] Fred H. Croom. *Basic Concepts of Algebraic Topology*. Springer–Verlag, 1978.
[Die98] Jean Dieudonné. *A History of Algebraic and Differential Topology 1900-1960*. Birkhäuser, 1998.
[ES52] Samuel Eilenberg and Norman Steenrod. *Foundations of Algebraic Topology*. Princeton University Press, 1952.
[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.
[Mas91] William S. Massey. *A Basic Course in Algebraic Topology*. Springer–Verlag, 1991.
[May99] J. Peter May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999.
[Sag94] Hans Sagan. *Space-Filling Curves*. Springer–Verlag, 1994.
[Spa66] Edwin Henry Spanier. *Algebraic Topology*. Springer–Verlag, 2nd edition, 1966.

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